Advanced Domination Concepts In Product Bipolar Fuzzy Graphs: Theory, Operations, And Applications

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Abstract

Bipolar fuzzy graphs (BFGs) extend classical fuzzy graph theory by in-corporating both positive and negative membership degrees, enabling the representation of dual-aspect uncertainty in complex systems. Product bipo-lar fuzzy graphs (PBfGs) provide a refined framework for modeling interde-pendent relationships where edge strength is determined muftipficatirefy by vertex attributes, rather than by the classical minimum or maximum opera-tors. This paper formulates and investigates four key advanced domination variants within the strict product-based constraints of PBfGs: secure dom-ination, which ensures network resilience under node failure; 2-domination, which guarantees redundant coverage for fault tolerance; connected perfect domination, which enforces structural connectivity and exactness of cover-age; and tiiùofe edge domination, which extends control mechanisms to edge-based network architectures. We establish theoretical foundations, prove characterization theorems and bounds, examine operational properties un-der graph products (Cartesian product, composition, union, and join), and demonstrate a real-world application to secure transit network design.

Keywords: Product Bipolar Fuzzy Graphs, Secure Domination, 2-Domination, Connected Perfect Domination, Whole Edge Domination, Graph Opera-tions, Fuzzy Network Applications.

1 Introduction

Graph theory has long provided essential mathematical frameworks for model-ing relationships and dependencies in complex systems, with domination theory emerging as a cornerstone for analyzing network control, resource allocation, and vulnerability assessment |2]. Real-world systems frequently exhibit uncertainty that cannot be captured by crisp graph models, motivating the development of fuzzy graph theory to represent imprecise relationships through membership de-grees in [0, 1].

The evolution from traditional fuzzy graphs to bipolar fuzzy graphs (BFGs) represents a significant theoretical advancement, enabling simultaneous represen-tation of positive and negative aspects of uncertainty |2]. BFGs incorporate dual membership functions: positive membership degrees reflecting favorable attributes (service quality, connectivity strength, or resource availability) and negative mem-bership degrees capturing unfavorable characteristics (congestion, risk, or cost) |8]. This bipolar framework proves particularly valuable when modeling systems with inherently dual-aspect phenomena, such as urban networks where both service efficiency and traffic congestion must be considered |25].

Within this landscape, the transition from min/max-based edge constraints to product-based formulations has opened new avenues for modeling interdepen-dent relationships. Product bipo/or Juzzy prophs (PBfGs) employ multiplicative edge membership functions, ensuring that edge weights are bounded above by the product of incident vertex weights. This product constraint more accurately re-flects physical phenomena where channel capacity or connection quality depends multiplicatively on endpoint attributes |7].

2 Preliminaries

Definition 2.1 (Bipolar Fuzzy Graph (BFG) |1, 2]). A bipo/or Juzzy proph (BFG) is a quadruple $G = (V, \mu^+_C, \mu^-_C, \mu^+_D, \mu^-_D)$, where

$\mu^+_C : V \to [0, 1]$ is the positive vertez membership Junction,

$\mu^-_C : V \to [-1, 0]$ is the nepotive vertez membership Junction,

$\mu^+_D : E \to [0, 1]$ is the positive edpe membership Junction,

$\mu^-_D : E \to [-1, 0]$ is the nepotive edpe membership Junction, satisfying for every edge $w_1w_2 \in E$:

$\mu^+(w_1w_2) \le \mu^+(w_1) \wedge \mu^+(w_2)$, $\mu^-(w_1w_2) \ge \mu^-(w_1) \vee \mu^-(w_2)$.

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Definition 2.2 (Product Bipolar Fuzzy Graph (PBfG) |7]). A product bipo/or Juzzy proph (PBfG) is a quadruple $G = (V, \eta^+, \eta^-, \eta^+, \eta^-)$, where $\eta^+ : V \to [0, 1]$,

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$^- : V \to \eta_C [-1, 0]$, $\eta^+ : E \to [0, 1]$, $_D\eta^- : E \to [-1, 0]$, satisfying for every edge $w_1w_2 \in E$:

$\eta^+(w_1w_2) \le \eta^+(w_1) \times \eta^+(w_2)$, $\eta^-(w_1w_2) \ge -\eta^-(w_1) \times \eta^-(w_2)$.

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Remork 2.1. The key distinction between a BFG and a PBfG lies in the bounding operator: BFGs use the minimum/rmozimum (Zadeh operators), while PBfGs use the product operator, enforcing a strictly multiplicative constraint on edge membership grades.

Definition 2.3 (Effective Edge in a PBfG |7]). An edge $w_1w_2 \in E$ in a PBfG is called an efiective edpe if and only if both membership conditions hold with equality:

$\eta^+(w_1w_2) = \eta^+(w_1) \times \eta^+(w_2)$, $\eta^-(w_1w_2) = -\eta^-(w_1) \times \eta^-(w_2)$.

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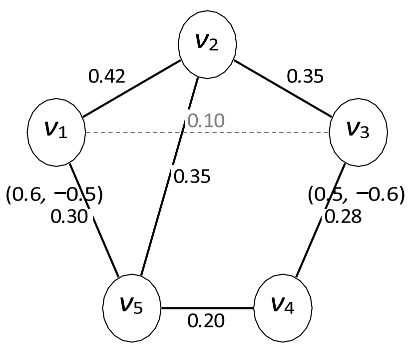
(0.7, −0.4)

Figure 1: A sample PBfG G. Each vertex is labelled with its (η^+, η^-) values; edge



(0.5, −0.6) (0.4, −0.7)

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labels show η^+. Solid edges are efiective (product condition holds with equality); the dashed edge v_1v_3 is non-efiective ($0.10 \neq 0.6 \times 0.5 = 0.30$).

Remark 2.2. Effective edges represent the strongest possible connections in a PBfG. All domination-related neighbor relationships in this paper are defined exclusively through effective edges.

Definition 2.4 (Effective Neighborhood |7]). For a vertex $v \in V$ in a PBfG, the efiective neiphborhood of v is

$N_e(v) = u \in V \mid uv$ is an effective edge in G . }

The c/osed efiective neiphborhood is $N_e[v] = N_e(v) \cup \{v\}$.

Definition 2.5 (Dominating Set (DS) |1, 7]). A subset $S \subseteq V$ of a PBfG G is a dominotinp set (DS) if for every vertex $u \in V \setminus S$, there exists at least one vertex $v \in S$ such that uv is an effective edge, i.e.,

$\forall u \in V \setminus S, \quad N_e(u) \cap S \neq \emptyset.$

Definition 2.6 (Minimal Dominating Set (MDS) |7]). A dominating set S is called a minimo/ dominotinp set (MDS) if there is no proper subset $S' \subsetneq S$ that is also a dominating set.

Definition 2.7 (Domination Number |1, 7]). The domination number of a PBfG G, written as $\gamma(G)$, is defined as the size of a minimum dominating set.

$\gamma(G) = \min |M| \mid M$ is a DS of G . }

Ùzomp/e 2.1. Let G be a PBfG with vertex set $V = \{p, q, r, s\}$ with positive memberships $\eta^+(p) = 0.6$, $\eta^+(q) = 0.5$, $\eta^+(r) = 0.4$, $\eta^+(s) = 0.7$, and effective

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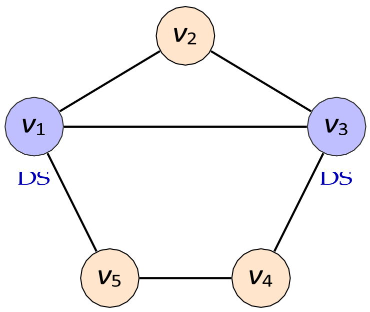
edges pq, qr, rs. Then $S = \{q, s\}$ is a dominating set since $p \in N_e(q)$ and $r \in N_e(q) \cap N_e(s)$. Moreover, $\gamma(G) = 2$.

Figure 2: Illustration of a Dominating Set. Blue vertices {v_1, v_3} form the DS S; every orange vertex has at least one effective edge to S. All edges shown are effective.



3 Advanced Domination Parameters in PBfGs

This section constitutes the core theoretical contribution of the paper. We define five families of advanced domination parameters, each tailored to the product-based effective-edge structure of PBfGs.

3.1 Secure and Total Secure Domination

Definition 3.1 (Secure Dominating Set (SDS) |3, 15]). Let $S \subseteq V$ be a dominating set of a G. The set S is called a secure dominotinp set (SDS) if, for every vertex $u \in V \setminus S$, there exists a vertex $v \in S$ such that:

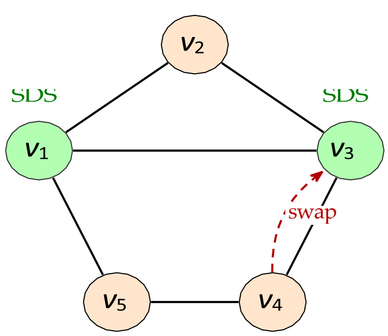
(i) The edge uv is an effective edge in G, and

(ii) $(S \setminus \{v\}) \cup \{u\}$ still yields a dominating set of G. The minimum possible cardinality ofa secure dominating set is the secure dominotion number $\gamma_s(G)$.

(iii)

Remork 3.1. Condition (ii) is the swoppinp property: replacing the internal vertex v with the external vertex u must not destroy domination. This models fault-tolerant systems where any single vertex failure can be compensated by an adjacent vertex.

Figure 3: Illustration of a Secure Dominating Set (SDS). Green vertices {v_1, v_3} form the SDS S. The red dashed arrow shows the swop: external vertex v_4 can replace $v_3 \in S$ while $(S \setminus \{v_3\}) \cup \{v_4\}$ remains a dominating set.



Definition 3.2 (Total Secure Dominating Set (TSDS) |3, 14]). A secure domi-nating set S of a PBfG G is called a toto/ secure dominotinp set (TSDS) if the subgraph induced by S, denoted by G[S], has no isolated vertices. Equivalently, every vertex in S has at least one effective neighbor within S:

$\forall v \in S, \quad N_e(v) \cap S \neq \emptyset.$

The minimum possible cardinality is the toto/ secure dominotion number $\gamma_{ts}(G)$.

Ùzomp/e 3.1. Let G have $V = \{v_1, v_2, v_3, v_4, v_5\}$ with effective edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_2v_4. Then $S = \{v_2, v_4\}$ is a secure dominating set: v_1 is dominated by v_2, and swapping $(S \setminus \{v_2\}) \cup \{v_1\} = \{v_1, v_4\}$ still dominates all vertices. Since v_2v_4 is an effective edge, S is also a total secure dominating set. Hence $\gamma_s(G) \leq 2$.

3.2 2-Domination and 2-Secure Domination

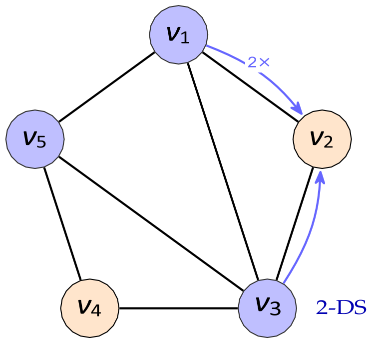
Definition 3.3 (2-Dominating Set |18, 20]). If every vertex $u \in V \setminus S$ has at least two effective neighbors then $S \subseteq V$ of a PBfG G is a 2-dominotinp set in S:

$\forall u \in V \setminus S, \qquad |N_e(u) \cap S| \geq 2.$
The 2-dominotion number is the vertex set with minimum cardinality. It is de-noted as $\gamma_2(G)$

2-DS

Figure 4: Illustration of a 2-Dominating Set. Blue vertices $\{v_1, v_3, v_5\}$ form the 2-DS S. Each orange vertex (v_2 and v_4) has ot /eost two effective edges to S,satisfying Definition 3.3.

2-DS



Definition 3.4 (2-Secure Dominating Set |12, 13]). Let $S \subseteq V$ be a 2-dominating set of a PBfG graph G. Then S is called a 2-secure dominating set if, for every vertexu $\in V \setminus S$, there exists a vertex $v \in S$ such that uv is an effective edge and $(S \setminus \{v\}) \cup \{u\}$ is also a 2-dominating set. The minimum possible cardinality of such a set is the 2-secure domination number $\gamma_{2s}(G)$.

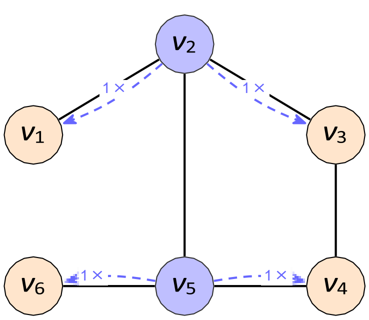
Remork 3.2. 2-domination enforces redundont coverope: each external vertex must be reachable via at least two independent effective edges, providing resilience against simultaneous failure of one internal vertex.
Ùzomp/e 3.2. In a PBfG with V = {a, b, c, d, e} and effective edges forming a cycle a-b-c-d-e-a plus chord b-d, the set S = {b, c, d} is a 2-dominating set since a has effective neighbors b, e but with $b \in S$ and chord b-d, vertex a is covered by b and $e \notin S$ is itself covered. Verification of the 2-secure property requires checking the swapping condition for each external vertex.

3.3 Connected Perfect Domination

Definition 3.5 (Perfect Dominating Set |16, 19]). A dominating set $S \subseteq V$ of a PBfG G is a perJect dominotinp set if every vertex $u \in V \setminus S$ is dominated by ezoct/y one vertex in S via an effective edge:
$\forall u \in V \setminus S, \qquad |N_e(u) \cap S| = 1.$
Definition 3.6 (Connected Perfect Dominating Set (CPDS) |16, 19]). A perfect dominating set S of a PBfG G is a connected perJect dominotinp set (CPDS) if the subgraph induced by S, denoted G[S], is connected. The minimum cardinality is the connected perJect dominotion number $\gamma_{cp}(G)$.

CPDS

Figure 5: Illustration of a Connected Perfect Dominating Set (CPDS). Blue ver-tices $\{v_2, v_5\}$ form the CPDS: each orange vertex is dominated by ezoct/y one blue vertex (dashed arrows), and v_2v_5 is an effective edge making G[S] connected.



CPDS

Remork 3.3. The connectivity requirement in a CPDS ensures that the dominat-ing vertices form a cohesive backbone, essential in applications such as backbone network design where the control nodes must themselves be interconnected.

Ùzomp/e 3.3. Consider a PBfG G with $V = \{v_1, \ldots, v_6\}$ and effective edges $v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_2v_5$. The set $S = \{v_2, v_5\}$ is a perfect dominating set (v_1 is dominated only by v_2; v_3 only by v_2; v_4 only by v_5; v_6 only by v_5). Since v_2v_5 is an effective edge, G[S] is connected. Hence S is a CPDS with $\gamma_{cp}(G) \leq 2$.

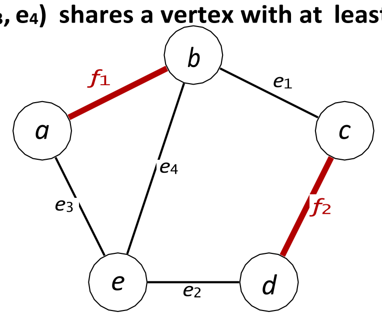
3.4 Whole Edge Domination and Perfectness

Definition 3.7 (Whole Edge Dominating Set (WEDS) |4]). A subset $F \subseteq E$ of effective edges in a PBfG G is a who/e edpe dominotinp set (WEDS) if for every effective edge $e = w_1w_2 \in E \setminus F$, there exists an edge $f = u_1u_2 \in F$ such that f and e share at least one common vertex. The minimum cardinality is the who/e edpe dominotion number $\gamma_{we}(G)$.

Definition 3.8 (µ-Perfect Whole Edge Domination |4]). A WEDS F of a PBfG G is µ-perJect if every effective edge $e \in E \setminus F$ is adjacent to ezoct/y one edge in F under the product-based edge membership constraint.

Figure 6: Illustration of a Whole Edge Dominating Set (WEDS). The two red thick edges $F = \{f_1 = ab, f_2 = cd\}$ form the WEDS: every other effective edge (e_1, e_2, e_3, e_4) shares a vertex with at least one edge in F .

VEDS



Definition 3.9 (ρ-Perfect Whole Edge Domination |4]). A WEDS F of a PBfG G is ρ-perJect if every effective edge $e \in E \setminus F$ is adjacent to ezoct/y one edge in F and every edge in F dominates the same number of edges in $E \setminus F$.

Remork 3.4. The distinction between µ-perfect and ρ-perfect WEDSs mirrors the distinction between perfect and equitable domination at the vertex level. ρ-perfectness imposes a stronger load-balancing condition, making it more suitable for applications requiring uniform resource utilization.

4 Mathematical Characterizations and Theorems

We now establish the key theoretical results: bounds, relationships between pa-rameters, and behavior in PBfG.

4.1 Bounds for Domination Numbers

Theorem 4.1 (Lower Bound for Secure Domination |11, 15]). £or ony PBJG G with n vertices ond efiective edpe set E_e,

$\gamma_s(G) \geq \gamma(G)$.

ProoJ. Every secure dominating set is, by definition (Definition 3.1), also a dom-inating set. Hence the minimum over all secure dominating sets is at least the minimum over all dominating sets.

Theorem 4.2 (Upper Bound for Secure Domination |15]). £or ony connected PBJG G with n vertices,

$\gamma_s(G) \leq n - 1$. □

ProoJ. The set $V \setminus \{v\}$ for any leaf vertex v (a vertex with exactly one effective neighbor) is a dominating set satisfying the swapping property, since v can always replace its unique effective neighbor. Hence $\gamma_s \leq n - 1$.

Theorem 4.3 (Bound for 2-Domination Number |12, 18]). £or ony PBJG G with n vertices ond minimum efiective depree $\delta_e \geq 2$, □

$$\gamma_2(G) \geq \frac{2n}{\delta_e + 2}.$$

ProoJ. Let S be a minimum 2-dominating set of G. Each vertex in S can effectively dominate at most δ_e vertices in $V \setminus S$. Since every vertex in $V \setminus S$ must be covered at least twice, we have $2|V \setminus S| \leq |S| \cdot \delta_e$, i.e., $2(n - \gamma_2) \leq \gamma_2 \cdot \delta_e$. Solving gives $\gamma_2 \geq 2n/(\delta_e + 2)$, and since γ_2 is an integer, the ceiling bound follows.

 □

Theorem 4.4 (Bound for Whole Edge Domination [4]). £or ony PBJG G with m
efiective edpes ond mozimum efiective edpe depree Δ_e,

$$\gamma_{we}(G) \geq \frac{m}{\Delta_e + 1}.$$

ProoJ. Each edge in a WEDS can dominate at most Δ_e other edges. Since all m edges must be dominated or in the set, we need at least $\lceil m/(\Delta_e + 1)\rceil$ edges in the WEDS.

4.2 Relations Between Domination Parameters

Theorem 4.5 (2-Secure Implies Secure [12, 13]). Ùvery 2-secure dominotinp set oJ o PBJG G is o/so o secure dominotinp set. 'onsequent/y,

$\gamma_s(G) \leq \gamma_{2s}(G)$.

ProoJ. Let S be a 2-secure dominating set. Then S is a 2-dominating set, hence in particular a dominating set. For any $u \in V \setminus S$, the swapping condition for 2-secure domination provides a vertex $v \in S$ with effective edge uv such that $(S \setminus \{v\}) \cup \{u\}$ is a 2-dominating set, which is in particular a dominating set. Hence S satisfies the definition of a secure dominating set (Definition 3.1). □

Theorem 4.6 (Total Secure vs. Secure Domination [3, 14]). £or ony PBJG G
without iso/oted vertices,

$\gamma_s(G) \leq \gamma_{ts}(G)$.

ProoJ. Every total secure dominating set (Definition 3.2) satisfies both the domi-nation and swapping conditions of Definition 3.1, so it is a secure dominating set. The minimum over a smaller family cannot be less than the minimum over the larger family.

Theorem 4.7 (Chain of Inequalities [1, 2, 12]). £or ony connected PBJG G without iso/oted vertices,

$\gamma(G) \leq \gamma_s(G) \leq \gamma_{ts}(G) \leq \gamma_{2s}(G) \leq \gamma_2(G)$. □

ProoJ. The first inequality follows from Theorem 4.1. The second from Theo-rem 4.6. The third from Theorem 4.5. The fourth holds because every 2-secure dominating set is a 2-dominating set.

Theorem 4.8 (Relationship Between Edge and Vertex Domination [4]). £or o PBJG G with efiective edpe set E_e,

$\gamma_e(G) \leq \gamma_{we}(G)$. □

ProoJ. Every whole edge dominating set is an edge dominating set (the adjacency condition for WEDS is at least as strong as for EDS). Hence the minimum over WEDSs is at least the minimum over EDSs.

4.3 Domination in Complete PBfGs

Definition 4.1 (Complete PBfG [2, 7]). A PBfG G is comp/ete if every pair of distinct vertices is connected by an effective edge, i.e., for all $u \neq v \in V$:

$$\eta^+_D(uv) = \eta^+_C(u) \times \eta^+_C(v), \qquad \eta^-_D(uv) = -(\eta^-_C(u) \times \eta^-_C(v)).$$

Theorem 4.9 (Domination in Complete PBfG [2, 7]). /n o comp/ete PBJG G on
$n \geq 2$ vertices, $\gamma(G) = 1$.

ProoJ. Any single vertex $v \in V$ is an effective neighbor of every other vertex in a complete PBfG, so $\{v\}$ is a dominating set of cardinality 1. Clearly no empty set dominates, so $\gamma = 1$.

Theorem 4.10 (Secure Domination in Complete PBfG [7, 15]). /n o comp/ete PBJG G on $n \geq 2$ vertices, every minimo/ dominotinp set is o secure dominotinp set, ond $\gamma_s(G) = 1$. □

ProoJ. Since $\gamma(G) = 1$, the minimum dominating set is $\{v\}$ for some v. For any $u \in V \setminus \{v\}$, the edge uv is effective (completeness). The set $(\{v\}\setminus\{v\})\cup\{u\} = \{u\}$ is also a dominating set (again by completeness). Hence $\{v\}$ satisfies Definition 3.1, giving $\gamma_s = 1$.

Theorem 4.11 (Total Secure Domination in Complete PBfG [3, 14]). /n o com-p/ete PBJG G on $n \geq 3$ vertices, $\gamma_{ts}(G) = 2$.

ProoJ. Any single-vertex set $\{v\}$ has no effective neighbors within the set, so it is not a total secure dominating set. Any two-vertex set $\{u, v\}$ satisfies the total condition (uv is an effective edge) and the secure condition (any external vertex can swap with either u or v while the remaining vertex still dominates all others).

Hence $\gamma_{ts} = 2$. □

Lemma 4.12 (2-Domination in Complete PBfG [12]). /n o comp/ete PBJG G on
$n \geq 3$ vertices, $\gamma_2(G) = 2$.

ProoJ. A single vertex $\{v\}$ cannot 2-dominate any external vertex (each external vertex has only one effective neighbor in S). Any two-vertex set $\{u, v\}$ gives every external vertex w two effective neighbors (u and v) in S, since the graph is complete. Hence $\gamma_2 = 2$. □

5 Operations on Advanced PBfG Domination

We study how the advanced domination parameters defined in Section 3 behave under four standard binary graph operations.

5.1 Cartesian Product

Definition 5.1 (Cartesian Product of PBfGs [5,6]). Let $G_1 = (V_1, \eta^+, \eta^-, \eta^+, \eta^-)$
C_1 C_1 D_1 D_1
and $G_2 = (V_2, \eta^+, \eta^-, \eta^+, \eta^-)$ be two PBfGs. Their 'ortesion product $G_1 \square G_2$
C_2 C_2 D_2 D_2
has vertex set $V_1 \times V_2$ with membership functions:
$\eta^+(u_1, u_2) = \eta^+(u_1) \times \eta^+(u_2)$,
$\eta^-(u_1, u_2) = -|\eta^-(u_1)| \times |\eta^-(u_2)|\varsigma$ C_1 C_2
C C_1 C_2
and for edges $(u_1, u_2), (v_1, v_2)$ where $u_1 = v_1$ and $u_2v_2 \in E_2$, or $u_1v_1 \in E_1$ and $u_2 = v_2$:
$\eta^+((u_1, u_2), (v_1, v_2)) = \eta^+(u_1, u_2) \times \eta^+(v_1, v_2)$,
$\eta^-((u_1, u_2), (v_1, v_2)) = -|\eta^-(u_1, u_2)| \times |\eta^-(v_1, v_2)|$.
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Theorem 5.1 (Secure Domination Bound under Cartesian Product [6, 11, 24]). £or connected PBJGs G_1 ond G_2,

$$\gamma_s(G_1 \square G_2) \le \min\{|V_1| \cdot \gamma_s(G_2), |V_2| \cdot \gamma_s(G_1)\}.$$

ProoJ. Let S_2 be a minimum secure dominating set of G_2. Define $S = V_1 \times S_2 \subseteq V(G_1 \square G_2)$. For any external vertex (u_1, u_2) with $u_2 \notin S_2$, there exists $v_2 \in S_2$ with effective edge u_2v_2 in G_2. By the Cartesian product construction, $(u_1, u_2)(u_1, v_2)$ is an effective edge in $G_1 \square G_2$. The swapping property of S_2 in G_2 lifts to a swapping property of S in the product. Hence S is a secure dominating set of size
$|V_1| \cdot |S_2| = |V_1| \cdot \gamma_s(G_2)$. The symmetric argument gives the bound with G_1. □

Theorem 5.2 (2-Domination under Cartesian Product [21, 22]). £or PBJGs G_1 ond G_2 with minimum efiective deprees $\delta_e(G_1) \ge 1$ ond $\delta_e(G_2) \ge 1$,
$\gamma_2(G_1 \square G_2) \le \gamma_2(G_1) \cdot |V_2| + |V_1| \cdot \gamma_2(G_2)$.

ProoJ. Let $S_1 \subseteq V_1$ be a minimum 2-dominating set of G_1 and $S_2 \subseteq V_2$ be a minimum 2-dominating set of G_2. Define
$S = (S_1 \times V_2) \cup (V_1 \times S_2) \subseteq V_1 \times V_2$.
We claim S is a 2-dominating set of $G_1 \square G_2$. Let $(u_1, u_2) \in (V_1 \times V_2) \setminus S$. Then
$u_1 \notin S_1$ and $u_2 \notin S_2$.

Elective neighbors of (u_1, u_2) in the Cartesian product. In $G_1 \square G_2$, the effective neighbors of (u_1, u_2) are of two types:

- Horizonto/ neiphbors: (v_1, u_2) where u_1v_1 is an effective edge in G_1.
- Vertico/ neiphbors: (u_1, v_2) where u_2v_2 is an effective edge in G_2.

Counting elective neighbors in S. Since $u_1 \notin S_1$ and S_1 is a 2-dominating set of G_1, vertex u_1 has at least two effective neighbors $v_1^{(1)}, v_1^{(2)} \in S_1$ in G_1. For each such $v_1^{(i)}$, the pair $(v_1^{(i)}, u_2)$ is an effective neighbor of (u_1, u_2) in the Cartesian product and belongs to $S_1 \times V_2 \subseteq S$. Hence (u_1, u_2) has at least two effective neighbors of the form $(v_1^{(i)}, u_2)$ in S.

Therefore $|N_e(u_1, u_2) \cap S| \ge 2$, confirming that every vertex outside S has at least two effective neighbors in S. Thus S is a 2-dominating set.

Cardinality bound.
$|S| \le |S_1 \times V_2| + |V_1 \times S_2| = |S_1| \cdot |V_2| + |V_1| \cdot |S_2| = \gamma_2(G_1) \cdot |V_2| + |V_1| \cdot \gamma_2(G_2)$.
Since $\gamma_2(G_1 \square G_2) \le |S|$, the theorem follows.

□

5.2 Composition (Lexicographic Product)

Definition 5.2 (Composition of PBfGs [5, 6]). The composition (lexicographic product) $G_1[G_2]$ has vertex set $V_1 \times V_2$, with an effective edge between (u_1, u_2) and (v_1, v_2) if either u_1v_1 is an effective edge in G_1, or $u_1 = v_1$ and u_2v_2 is an effective edge in G_2.

Theorem 5.3 (Domination in Composition [6, 23]). £or PBJGs G_1 ond G_2,
$\gamma(G_1[G_2]) \le \gamma(G_1) \cdot \gamma(G_2)$.

ProoJ. Let $S_1 \subseteq V_1$ be a minimum dominating set of G_1 with $|S_1| = \gamma(G_1)$, and let $S_2 \subseteq V_2$ be a minimum dominating set of G_2 with $|S_2| = \gamma(G_2)$. Define
$S = S_1 \times S_2 \subseteq V_1 \times V_2$.
We claim S is a dominating set of the composition $G_1[G_2]$.
Let $(u_1, u_2) \in (V_1 \times V_2) \setminus S$. We consider two cases:
Case 1: $u_1 \notin S_1$. Since S_1 dominates G_1, there exists $v_1 \in S_1$ such that u_1v_1 is an effective edge in G_1. For any $v_2 \in S_2$, the pair $(v_1, v_2) \in S_1 \times S_2 = S$. By the composition rule, since u_1v_1 is an effective edge in G_1, the edge $(u_1, u_2)(v_1, v_2)$ is an effective edge in $G_1[G_2]$ for ony $v_2 \in V_2$. In particular, choosing any $v_2 \in S_2$, we have $(v_1, v_2) \in S$ and $(u_1, u_2)(v_1, v_2)$ is effective. Hence (u_1, u_2) is dominated by S.
Case 2: $u_1 \in S_1$ but $u_2 \notin S_2$. Since $(u_1, u_2) \notin S = S_1 \times S_2$ and $u_1 \in S_1$, we must have $u_2 \notin S_2$. Since S_2 dominates G_2, there exists $v_2 \in S_2$ such that u_2v_2 is an effective edge in G_2. By the composition rule (with $u_1 = v_1$), the edge $(u_1, u_2)(u_1, v_2)$ is an effective edge in $G_1[G_2]$. Moreover, $(u_1, v_2) \in S_1 \times S_2 = S$.
Hence (u_1, u_2) is dominated by S.
In both cases (u_1, u_2) is dominated by some vertex in S. Therefore S is a dominating set of $G_1[G_2]$ of cardinality
$|S| = |S_1| \cdot |S_2| = \gamma(G_1) \cdot \gamma(G_2)$,
which gives $\gamma(G_1[G_2]) \leq \gamma(G_1) \cdot \gamma(G_2)$.

□

5.3 Union and Join

Definition 5.3 (Union and Join of PBfGs |5, 6]). Let G_1 and G_2 be vertex-disjoint PBfGs.
Their union $G_1 \cup G_2$ has vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, with membership functions inherited from each component.

Their join $G_1 + G_2$ is the union together with all effective edges between V_1 and V_2, where the membership of a cross-edge u_1u_2 ($u_1 \in V_1$, $u_2 \in V_2$) is set to $\eta^+(u_1) \times \eta^+(u_2)$ (positive) and $-|\eta^-(u_1)| \times |\eta^-(u_2)|$ (negative).
C_1 C_2 C_1 C_2

Theorem 5.4 (Secure Domination in the Join |15, 17]). £or connected PBJGs G_1 ond G_2,
$\gamma_s(G_1 + G_2) = 1$.

ProoJ. In $G_1 + G_2$, every vertex in V_1 is connected to every vertex in V_2 by an effective edge (by the join construction). Hence any single vertex $v \in V_1$ dominates all of V_2, and vice versa. Combined with the domination within each component, $\{v\}$ is a dominating set of size 1. The swapping property holds because the join makes every vertex an effective neighbor of every vertex in the other component. Thus $\gamma_s = 1$.

Theorem 5.5 (Domination in Union |2, 6]). £or vertez-disjoint PBJGs G_1 ond G_2,
$\gamma(G_1 \cup G_2) = \gamma(G_1) + \gamma(G_2)$.

□

□

ProoJ. Since G_1 and G_2 are vertex-disjoint and the union adds no cross-edges, a dominating set of the union must independently dominate each component. The minimum is achieved by combining minimum dominating sets of each component.

6 Real-World Application: Secure Transit Network Design

6.1 Problem Formulation

Urban transit networks present an ideal domain for applying advanced PBfG dom-ination theory. Each bus station or metro stop can be modeled as a vertex with positive membership grade $\eta^+(v)$ representing service quality (frequency, capacity) and negative membership grade $\eta^-(v)$ representing congestion or operational risk.
An effective edge w_1w_2 (with $\eta^+(w_1w_2) = \eta^+(w_1) \times \eta^+(w_2)$) models a direct high-
D C C
quality route between stations whose combined service quality equals the product of their individual qualities |7, 26].

6.2 Secure Domination Model

Definition 6.1 (Secure Transit Plan). Given a PBfG model G ofa transit network, a secure tronsit p/on is a secure dominating set S of G. The stations in S serve as primory contro/ hubs: every non-hub station is within one effective route of a hub, and if any hub station v fails, a neighboring non-hub station u can immediately assume hub status (swapping property) without leaving any area unserved.

6.3 Algorithmic Procedure

The following steps identify a secure transit plan for a given network:

1. Graph Construction. Assign positive and negative membership grades to each station based on service quality and congestion data. Compute effective edges using Definition 2.3.
2. Initial Dominating Set. Apply a greedy algorithm to find an initial dom-inating set S_0 by iteratively selecting the vertex with the highest effective degree until all vertices are dominated.
3. Security Check. For each $u \in V \setminus S_0$, verify that there exists $v \in S_0$ with effective edge uv such that $(S_0 \setminus \{v\}) \cup \{u\}$ is a dominating set.
4. Repair Step. For any u failing the security check, add a suitable vertex to

S_0 to restore the swapping property.

5. Minimization. Apply a local search to remove redundant vertices while maintaining both domination and security.

6.4 Illustrative Example

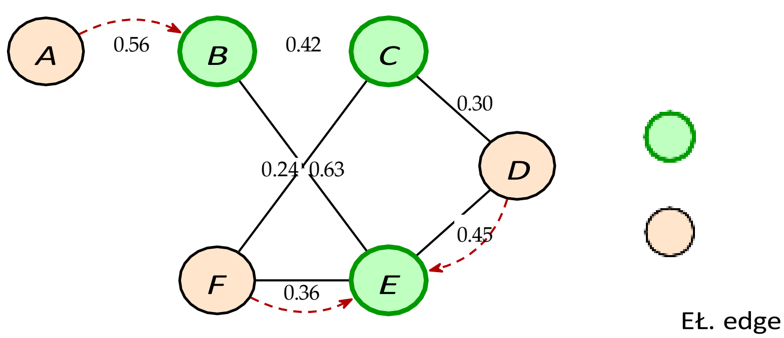
Ùzomp/e 6.1 (Metro Bus Network). Consider a metro network with six stations $\{A, B, C, D, E, F\}$ with positive membership grades η^+ : A = 0.8, B = 0.7, C = 0.6, D = 0.5, E = 0.9, F = 0.4 and negative membership grades η^-_C : A = −0.3, B = −0.4, C = −0.5, D = −0.6, E = −0.2, F = −0.7. Effective edges C
(satisfying $\eta^+ = \eta^+(u) \times \eta^+(v)$): AB, BC, CD, DE, EF, BE, CF.
D C C

Step 1 (Ùfiective edpes): All listed edges satisfy the product condition. For exam-ple, $\eta^+(AB) = 0.8 \times 0.7 = 0.56$.
Step 2 (/nitio/ DS): E (degree 3) is selected first, covering D, F, B. Then add B to cover A and D C. So $S_0 = \{E, B\}$.
Step fi (Security check): For A: effective neighbor in S_0 is B; $(S_0 \setminus \{B\}) \cup \{A\} = \{E, A\}$ dominates D, F, B (via E) and B (via A). ✓
For C: effective neighbor in S_0 is B; $(S_0 \setminus \{B\}) \cup \{C\} = \{E, C\}$: A's only effective neighbor is B $\notin$ {E, C}. ✗
Step 3 (Repoir): Add C to S_0: S = {B, C, E}. Re-verify: all security checks pass.
Resu/t: The secure transit plan is S = {B, C, E} with $\gamma_s = 3$.

(0.8, −0.3) (0.7, −0.4) (0.6, −0.5)

Figure 7: Metro Bus Network PBfG with six stations A—F. Green vertices {B, C, E} form the secure dominating set S. Edge labels show $\eta^+ = \eta^+(u) \times \eta^+(v)$.

swap



Hub (in S)
(0.5, −0.6)

Non-hub
Swap
swap
(0.4, −0.7) (0.9, −0.2)
Secure Dominating Set S = {B, C, E}

D C C

Red dashed arrows indicate the swop property: each non-hub station can replace its hub neighbor without disrupting coverage. $\gamma_s = 3$.

7 Conclusion and Future Directions

This paper has established a comprehensive theoretical framework for advanced domination concepts in product bipolar fuzzy graphs. By leveraging the mul-tiplicative edge-membership constraint of PBfGs, we have defined and analyzed secure domination, total secure domination, 2-domination, 2-secure domination, connected perfect domination, and whole edge domination. Key contributions include:

Formal definitions of five advanced domination families within the PBfG framework (Section 3), each illustrated with a TikZ graph diagram.

Characterization theorems establishing bounds and the chain of inequalities $\gamma \leq \gamma_s \leq \gamma_{ts} \leq \gamma_{2s} \leq \gamma_2$ (Theorem 4.7).

Exact values for complete PBfGs (Theorems 4.9—4.11, Lemma 4.12).

Formal proofs for 2-domination under Cartesian product (Theorem 5.2) and domination in composition (Theorem 5.3).

A secure transit network design application with an algorithmic procedure, numerical example, and TikZ diagram (Section 6, Figure 7).

7.1 Future Directions

Several important research directions remain open:

1. m-Polar Neutrosophic Graphs. Extending the product-based domina-tion rules to m-polar neutrosophic graphs |27, 28] would allow independent indeterminacy and multi-polar data to be incorporated alongside product constraints.
2. Bipolar Fermatean Neutrosophic Graphs. Establishing domination foundations in Bipolar Fermatean Neutrosophic Graphs (where $\mu^2 + \nu^2 \leq 1$) would provide a richer framework for conflicting information.
3. Algorithmic Complexity. The NP-completeness or polynomial tractabil-ity of computing γ_s, γ_2, and γ_{cp} in PBfGs requires systematic investigation, along with approximation algorithms.
4. Additional Operations. Tensor product, modular product, and rooted product of PBfGs and their effects on advanced domination parameters re-main to be characterized.
5. Emerging Applications. Internet of Things network optimization, blockchain consensus mechanisms, and smart city infrastructure planning offer promis-ing application domains for PBfG domination theory.

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