An operator approach to the ω-transfer correspondence between Appell and Central ω-Sheffer Families

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Abstract

Let δ_ω denote the central difference operator of step ω. The polynomial sequences that are Sheffer relative to $\omega^{-1}\delta_\omega$ the central ω-Sheffer class have been studied through their generating functions, but their relation to the classical Appell class has not been made explicit. We construct a single linear operator T_ω on the space of polynomials which carries every Appell sequence to its central ω-Sheffer partner, and which depends on the step alone and not on the invertible series defining the sequence. The operator is characterised as the unique isomorphism conjugating the derivative into $\omega^{-1}\delta_\omega$, and its matrix is that of the ω-central factorial numbers. We then solve the connection problem between the two classes. The connection matrix is shown to be a conjugation, $C(\omega) = \alpha\,\beta_\omega\,\alpha^{-1}$, and this is the feature specific to the central setting to be an even polynomial deformation of the identity: every entry lies in $\mathbb{K}[\omega^2]$ with $\deg_\omega c_{n,m} \leq n - m$, and $C(0) = I$. For the basic sequence the matrix has checkerboard support, $c_{n,m} = 0$ whenever $n \not\equiv m \pmod 2$. Exploiting the deformation structure we obtain the first-order expansion

$$s_n^{[\omega]}(x) = A_n(x) - \frac{\omega^2}{24}\, n(n-1)(n-2)\, x\, A_{n-3}(x) + O(\omega^4),$$

valid for every Appell sequence, together with a determinantal inversion formula. Two families are worked out completely — the ω-central factorials and the ω-central Bernoulli polynomials with full symbolic verification.

Keywords: Appell polynomials; central factorial numbers; ω-Sheffer sequences; central difference operator; connection coefficients; umbral operator.

1. Introduction

The classical Appell class [1] consists of the polynomial sequences $\{A_n\}$ satisfying the equation $A'_n = nA_{n-1}$, and contains the Bernoulli, Euler, Genocchi and Hermite families [6, 7]. Replacing the derivative by a delta operator produces the Sheffer class [2, 3], and different choices of delta operator produce structurally different discrete theories. The forward difference Δ_h yields the Δ_h-Appell and Δ_h-Sheffer families that have received considerable recent attention [13, 16]; the backward difference ∇_h yields a conjugate theory; and the central difference of step ω,

$$\delta_\omega f(x) = f(x + \omega/2) - f(x - \omega/2),$$

yields a third, whose basic sequence is the ω-central factorial and whose transition numbers are the central factorial numbers [8]. The elementary theory of these difference operators is classical [9, 10].

The central case is distinguished from the one-sided cases by a symmetry: δ_ω is invariant under the reflection $\omega \mapsto -\omega$ combined with a sign change, and the delta series associated with it is an odd function of the indeterminate. This parity has structural consequences which the forward and backward theories do not possess, and which appear to have gone unexploited. The present paper is organised around them.

Our approach is not the group-theoretic one. That the Appell sequences form a normal subgroup of the Sheffer group under umbral composition is classical [3, 5], and it is not the question addressed here. Instead we ask a constructive question: given the Appell class and given ω, is there an explicit operator carrying one class onto the other, and if so what does it do to the connection coefficients? The answers are, respectively, yes — a single operator T_ω independent of the invertible series — and: the connection matrix becomes an even polynomial deformation of the identity, with a perturbative expansion in ω^2 whose leading correction we compute in closed form for every Appell sequence simultaneously.

Relation to the classical theory: The general correspondence between Appell and Sheffer sequences via an umbral operator, and the general solution of the connection-constants problem for two Sheffer sequences, are classical and are due to Rota, Roman and their collaborators [3, 4, 5]; see also the surveys [18] and [17], and, for the state of Sheffer theory generally, [14]. We recall them in the form needed and claim no priority for them. What is new here is their specialisation to the central ω-setting and, in particular, the consequences of parity: Theorem 4.4 ($C(\omega) \in \mathbb{K}[\omega^2]$ with $C(0) = I$), Corollary 4.6 (checkerboard support), Theorem 5.1 (the universal ω^2 correction), and the worked structure of Sections 6 and 7. These have no counterpart in the forward- or backward-difference theories, where the associated delta series is not odd and no such deformation structure exists.

Section 2 fixes notation. Section 3 constructs the transfer operator. Section 4 solves the connection problem and establishes the parity theorems. Section 5 derives the perturbative expansion. Section 6 gives the determinantal inversion. Section 7 works two examples in full. Section 8 concludes.

2. Preliminaries

2.1 Formal series

We follow the conventions of [5] and [3]. Let $\mathbb{K}$ be a field of characteristic zero containing ω as a transcendental parameter, and let $\mathcal{F} = \mathbb{K}[[t]]$ carry the exponential normalisation $f(t) = \sum_k f_k\, t^k/k!$, so that

$$f(t)g(t) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k} f_k g_{n-k}\right)\frac{t^n}{n!}.$$

A series is invertible if $f(0) \neq 0$ and a delta series if $f(0) = 0$, $f'(0) \neq 0$. Write $\mathcal{P} = \mathbb{K}[x]$.

Definition 2.1. For invertible a, the Appell sequence generated by a is defined by

$$a(t)e^{xt} = \sum_{n\geq 0} A_n\,(x)\frac{t^n}{n!}, \qquad \text{whence} \qquad A_n(x) = \sum_{k=0}^{n}\binom{n}{k} a_{n-k}x^k,$$

and satisfies ${A'}_n = nA_{n-1}$.

2.2 The central difference operator and its delta series

Definition 2.2. For $\omega \neq 0$ the central difference operator and the associated normalised operator are

$$\delta_\omega f(x) = f(x+\omega/2) - f(x-\omega/2), \qquad D_\omega := \frac{1}{\omega}\,\delta_\omega.$$

Lemma 2.3. Let

$$\varphi_\omega(t) := \frac{2}{\omega}\sinh^{-1}\left(\frac{\omega t}{2}\right).$$

Then φ_ω is a delta series with ${\varphi'}_\omega(0) = 1$, it is an odd function of t, its coefficients are polynomials in ω^2, and

$$D_\omega\, e^{x\varphi_\omega(t)} = t\, e^{x\varphi_\omega(t)}.$$

Proof. Applying δ_ω in the variable x,

$$\delta_\omega e^{x\varphi_\omega(t)} = \left(e^{\omega\varphi_\omega(t)/2} - e^{-\omega\varphi_\omega(t)/2}\right)e^{x\varphi_\omega(t)} = 2\sinh\left(\omega/2\,\varphi_\omega(t)\right)e^{x\varphi_\omega(t)}.$$

By (2.4), $\omega/2\,\varphi_\omega(t) = \sinh^{-1}(\omega t/2)$, so the bracket equals $2\cdot\omega t/2 = \omega t$; dividing by ω gives (2.5). Since $\sinh^{-1}$ is odd with $\sinh^{-1}(u) = u - u^3/6 + 3u^5/40 - \cdots$, substitution gives

$$\varphi_\omega(t) = t - \frac{\omega^2}{24}t^3 + \frac{3\omega^4}{640}t^5 - \cdots,$$

which is odd in t, has $\varphi'_\omega(0) = 1$, and whose coefficient of t^{2j+1} is a constant multiple of ω^{2j}. Expansion (2.6) is the source of every parity result below, and it is worth isolating the two features it encodes: only odd powers of t occur, and only even powers of ω occur.

Definition 2.4. The ω-**central factorials** $\{x^{[n,\omega]}\}$ [8] are defined by

$$e^{x\varphi_\omega(t)} = \sum_{n\geq 0} x^{[n,\omega]} \frac{t^n}{n!}.$$

By (2.5) and comparison of coefficients, $D_\omega x^{[n,\omega]} = n\, x^{[n-1,\omega]}$, so $\{x^{[n,\omega]}\}$ is the basic sequence of D_ω; it is of binomial type. Direct expansion of (2.7) gives

$$x^{[0,\omega]} = 1, \quad x^{[1,\omega]} = x, \quad x^{[2,\omega]} = x^2, \quad x^{[3,\omega]} = x(x^2 - \omega^2/4), \quad x^{[4,\omega]} = x^2(x^2 - \omega^2),$$

in agreement with the classical product form $x^{[n,\omega]} = x\prod_{j=1}^{n-1}\big(x + \omega(n/2 - j)\big)$.

Definition 2.5. For invertible a, the **central** ω-**Sheffer sequence** for a is defined by

$$a(t)\, e^{x\varphi_\omega(t)} = \sum_{n\geq 0} s_n^{[\omega]}(x) \frac{t^n}{n!}.$$

Because a does not involve x, relation (2.5) gives at once

$$D_\omega\, s_n^{[\omega]}(x) = n\, s_{n-1}^{[\omega]}(x), \qquad \text{equivalently} \qquad \delta_\omega s_n^{[\omega]}(x) = n\,\omega\, s_{n-1}^{[\omega]}(x).$$

Setting $\omega \to 0$ in (2.4) gives $\varphi_0(t) = t$, so (2.9) degenerates to (2.2) and $s_n^{[0]} = A_n$.

2.3 Coefficient matrices

Write $A_n(x) = \sum_k \alpha_{n,k}\, x^k$ and $x^{[n,\omega]} = \sum_k \beta_{n,k}(\omega) x^k$, and let α, β_ω denote the corresponding infinite lower-triangular matrices. By (2.2), $\alpha_{n,k} = \binom{n}{k} a_{n-k}$ and $\alpha_{n,n} = a_0 \neq 0$. The entries $\beta_{n,k}(\omega)$ are, up to normalisation, the **central factorial numbers of the first kind** with step ω [8, 7]; by (2.6) they satisfy

$$\beta_{n,n}(\omega) = 1, \qquad \beta_{n,k}(\omega) = 0 \ \text{ whenever } n \not\equiv k \pmod 2,$$

the second assertion because $e^{x\varphi_\omega(t)} = \sum_m x^m\, \varphi_\omega(t)^m/m!$ and $\varphi_\omega(t)^m$ contains only powers t^j with $j \equiv m \pmod 2$. Both matrices are invertible in the algebra of infinite lower-triangular matrices.

3. The ω-Transfer Operator

Definition 3.1. The ω-**transfer operator** $T_\omega : \mathcal{P} \to \mathcal{P}$ is the linear map determined by

$$T_\omega(x^n) = x^{[n,\omega]}, \qquad n \geq 0.$$

Since $\{x^{[n,\omega]}\}$ is a polynomial sequence, T_ω maps a basis to a basis and is a linear isomorphism restricting to each $\mathcal{P}_n$; its matrix is β_ω. Note that T_ω depends on ω and on nothing else, and that $T_0 = I$ by (2.6).

Theorem 3.2 (Transfer correspondence). Let $\{A_n\}$ be the Appell sequence generated by a and $\{s_n^{[\omega]}\}$ the central ω-Sheffer sequence for the same a. Then

$$s_n^{[\omega]}(x) = T_\omega(A_n)(x) = \sum_{k=0}^{n} \binom{n}{k} a_{n-k}\, x^{[k,\omega]}.$$

Proof. Substituting (2.7) into (2.9) and applying the Cauchy product (2.1),

$$\sum_{n\geq 0} s_n^{[\omega]}(x) \frac{t^n}{n!} = \left(\sum_m a_m \frac{t^m}{m!}\right)\left(\sum_k x^{[k,\omega]} \frac{t^k}{k!}\right) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} \binom{n}{k} a_{n-k} x^{[k,\omega]}\right)\frac{t^n}{n!},$$

and comparison of coefficients gives the second equality. Applying T_ω to (2.2) and using (3.1) with linearity gives the first. ▫

Corollary 3.3. For fixed ω, the assignment $\{A_n\} \mapsto \{T_\omega A_n\}$ is a bijection from the Appell class onto the central ω-Sheffer class, preserving the generating invertible series. Its inverse is induced by T_ω^{-1}, whose matrix is β_ω^{-1}, the matrix of central factorial numbers of the second kind [8].

Proof. Surjectivity and well-definedness are Theorem 3.2; injectivity follows from that of T_ω. The invertible series is recovered by evaluating (2.9) at $x = 0$, since $x^{[k,\omega]}$ vanishes at $x = 0$ for $k \geq 1$: $a(t) = \sum_n s_n^{[\omega]}(0)t^n/n!$. ▫

Theorem 3.4 (Intertwining). As operators on $\mathcal{P}$,

$$D_\omega\, T_\omega = T_\omega\, D, \qquad \text{equivalently} \qquad D_\omega = T_\omega\, D\, T_\omega^{-1},$$

and T_ω is the unique linear isomorphism satisfying (3.3) together with $T_\omega(1) = 1$ and $(T_\omega x^n)(0) = 0$ for $n \geq 1$.

Proof: Both sides are linear, so it suffices to check on monomials. By (3.1) and Definition 2.4,

$$D_\omega T_\omega x^n = D_\omega x^{[n,\omega]} = n\, x^{[n-1,\omega]} = T_\omega(nx^{n-1}) = T_\omega D x^n,$$

with both sides zero at $n = 0$. For uniqueness, let T satisfy the hypotheses and set $p_n = T(x^n)$. Then $D_\omega p_n = np_{n-1}$ with $p_0 = 1$ and $p_n(0) = 0$ for $n \geq 1$; since D_ω lowers degree by exactly one, $\{p_n\}$ is a polynomial sequence obeying the defining relations of the basic sequence of D_ω, which is unique. Hence $p_n = x^{[n,\omega]}$ and $T = T_\omega$. ▫

Relation (3.3) gives an immediate second proof of (2.10):

$$D_\omega s_n^{[\omega]} = D_\omega T_\omega A_n = T_\omega D A_n = T_\omega(nA_{n-1}) = n\, s_{n-1}^{[\omega]}.$$

Corollary 3.5 (Transfer of operational identities). Let P be a linear form with coefficients in $\mathbb{K}$. If the Appell sequence satisfies $P\left(D^{j_0}A_{n_0}, \dots, D^{j_r}A_{n_r}\right) = 0$ identically, then

$$P\left(D_\omega^{j_0} s_{n_0}^{[\omega]}, \dots, D_\omega^{j_r} s_{n_r}^{[\omega]}\right) = 0.$$

Proof. Iterating (3.3) gives $D_\omega^j T_\omega = T_\omega D^j$; apply T_ω to the hypothesis and use linearity with $T_\omega A_n = s_n^{[\omega]}$. ▫

In particular $D^j A_n = \frac{n!}{(n-j)!} A_{n-j}$ transfers to

$$\delta_\omega^j\, s_n^{[\omega]}(x) = \frac{n!}{(n-j)!}\, \omega^j\, s_{n-j}^{[\omega]}(x), \qquad 0 \leq j \leq n,$$

with $\delta_\omega^j s_n^{[\omega]} \equiv 0$ for $j > n$.

Remark 3.6. By (3.3) the pair $(\mathbf{D}, \mathbf{D_\omega})$ is conjugate, and the monomiality apparatus of [11, 15] transfers accordingly: so if $\left(\widehat{\mathbf{M}}, \mathbf{D}\right)$ are multiplicative and derivative operators for $\{\mathbf{A_n}\}$ then $\left(\mathbf{T_\omega} \widehat{\mathbf{M}} \mathbf{T_\omega^{-1}}, \mathbf{D_\omega}\right)$ are such a pair for $\{\mathbf{s_n^{[\omega]}}\}$, and the Weyl relation $[\,\cdot\,,\cdot\,] = \mathbf{I}$ is preserved by conjugation. The quasi-monomiality of the central $\boldsymbol{\omega}$-Sheffer sequence is therefore inherited from its Appell partner without separate computation.

4. The Connection Problem and the Parity Structure

The problem solved in this section is the connection-constants problem of [5] in the special case in which one of the two sequences is Appell; see also [17, 18]. Since $\{A_m\}$ is a basis of $\mathcal{P}$ and $\deg s_n^{[\omega]} = n$, there are unique scalars $c_{n,m}(\omega)$, vanishing for $m > n$, with

$$s_n^{[\omega]}(x) = \sum_{m=0}^{n} c_{n,m}(\omega)\, A_m(x).$$

We call $C(\omega) = \left(c_{n,m}(\omega)\right)$ the ω-**connection matrix**.

4.1 Conjugation structure

Theorem 4.1. With α and β_ω as in Section 2.3,

$$C(\omega) = \alpha\, \beta_\omega\, \alpha^{-1}.$$

In the language of exponential Riordan arrays [19], relation (4.2) exhibits the connection matrix as a conjugate of the array of the basic sequence.

Proof. Let $X = (1, x, x^2, \dots)^T$, so that $A = \alpha X$ and $x^{[\cdot,\omega]} = \beta_\omega X$; all products are of infinite lower-triangular matrices and are defined entrywise by finite sums. By Theorem 3.2 the vector of ω-Sheffer polynomials is $s = \alpha\, \beta_\omega X$, while (4.1) reads $s = CA = C\alpha X$. The components of X are linearly independent over $\mathbb{K}$, so $\alpha\beta_\omega = C\alpha$, and α is invertible. ▫

Corollary 4.2. $C(\omega)$ is lower triangular with unit diagonal, $c_{n,n}(\omega) = 1$ for all n and all ω.

Proof. The diagonal of a product of triangular matrices is the product of the diagonals, so by (2.11), $c_{n,n} = a_0 \cdot 1 \cdot a_0^{-1} = 1$. ▫

Corollary 4.3. For fixed ω the connection matrices arising from different Appell sequences are mutually similar, each being conjugate to β_ω. The difficulty of the connection problem is therefore a property of the step alone.

4.2 Closed form and the parity theorems

Theorem 4.4. For $0 \le m \le n$,

$$c_{n,m}(\omega) = \frac{n!}{m!}\, [t^n] \left\{ \frac{a(t)}{a(\varphi_\omega(t))}\, \varphi_\omega(t)^m \right\}.$$

Moreover, if a does not depend on ω, then

$$c_{n,m}(\omega) \in \mathbb{K}[\omega^2], \qquad \deg_\omega c_{n,m} \le n - m, \qquad C(0) = I.$$

Proof. Since φ_ω is a delta series and a is invertible, $a(\varphi_\omega(t))$ is invertible and the quotient is legitimate. Replacing t by $\varphi_\omega(t)$ in (2.2),

$$a(\varphi_\omega(t)) e^{x\varphi_\omega(t)} = \sum_{m\ge0} A_m(x) \frac{\varphi_\omega(t)^m}{m!},$$

the substitution being legitimate because $\varphi_\omega(0) = 0$. Dividing by $a(\varphi_\omega(t))$, multiplying by $a(t)$ and using (2.9),

$$\sum_{n\ge0} s_n^{[\omega]}(x) \frac{t^n}{n!} = \frac{a(t)}{a(\varphi_\omega(t))} \sum_{m\ge0} A_m(x) \frac{\varphi_\omega(t)^m}{m!}.$$

Comparing coefficients of $t^n/n!$ gives (4.3); the sum terminates at $m = n$ because $\varphi_\omega(t)^m = O(t^m)$.

For (4.4), recall from (2.6) that every coefficient of φ_ω is a monomial in ω^2. Hence $\varphi_\omega(t)^m$ has coefficients in $\mathbb{K}[\omega^2]$, and so does the composition $a(\varphi_\omega(t))$, being a $\mathbb{K}$-linear combination of powers of φ_ω. Since $a(\varphi_\omega(0)) = a(0) \ne 0$, the inverse of $a(\varphi_\omega(t))$ has coefficients in $\mathbb{K}[\omega^2]$ as well, by the triangular recursion determining them. Therefore each coefficient extracted in (4.3) lies in $\mathbb{K}[\omega^2]$.

For the degree bound, observe from (2.6) that the coefficient of t^{2j+1} in φ_ω carries ω^{2j}; that is, each excess power of t beyond the first is accompanied by at most one power of ω. Consequently, extracting t^n from a product whose total t-order begins at m can produce at most ω^{n-m}.

Finally, $\varphi_0(t) = t$ gives $a(t)/a(\varphi_0(t)) = 1$ and $\varphi_0(t)^m = t^m$, so $c_{n,m}(0) = \frac{n!}{m!}[t^n]t^m = \delta_{n,m}$. ▫

Statement (4.4) is the structural core of the paper. It says that the passage from the Appell class to the central ω-Sheffer class is not an arbitrary change of basis but an **even polynomial deformation of the identity**, controlled by the single parameter ω^2. Neither the forward-difference nor the backward-difference theory has this property, because the delta series $h^{-1}\log(1 + ht)$ and $-h^{-1}\log(1 - ht)$ associated with Δ_h and ∇_h are not odd, and their coefficients carry odd powers of the step.

Corollary 4.5 (Consistency). Taking $m = n$ in (4.3), the quotient has value 1 at $t = 0$ and $\varphi_\omega(t)^n = t^n + O(t^{n+2})$, so $c_{n,n} = 1$, recovering Corollary 4.2.

Corollary 4.6 (Checkerboard support). If $a \equiv 1$ — that is, for the basic sequence $\{x^{[n,\omega]}\}$ expanded in the monomial basis — then

$$c_{n,m}(\omega) = 0 \qquad \text{whenever } n \not\equiv m \pmod 2.$$

More generally the same holds whenever $a(t)/a(\varphi_\omega(t))$ is an even function of t.

Proof. If the quotient in (4.3) is even in t and φ_ω is odd, then $\varphi_\omega(t)^m$ has parity m, so the product has parity m and its coefficient of t^n vanishes unless $n \equiv m \pmod 2$. For $a \equiv 1$ the quotient is identically 1. ▫

Remark. Corollary 4.6 is the connection-coefficient form of the classical vanishing of the central factorial numbers for $n - k$ odd [8], and the general statement (4.4) shows how that vanishing degrades under a non-even invertible series: parity in t is lost, but parity in ω survives.

5. The Perturbative Expansion

Because $C(\omega) = I + O(\omega^2)$ by Theorem 4.4, each central ω-Sheffer polynomial is a perturbation of its Appell partner. The leading correction can be computed once and for all, for every Appell sequence simultaneously.

Theorem 5.1. For every Appell sequence $\{A_n\}$ and the corresponding central ω-Sheffer sequence,

$$s_n^{[\omega]}(x) = A_n(x) - \frac{\omega^2}{24}\, n(n-1)(n-2)\; x\, A_{n-3}(x) + O(\omega^4),$$

with the convention $A_j \equiv 0$ for $j < 0$; in particular $s_n^{[\omega]} = A_n + O(\omega^4)$ for $n \le 2$.

Proof. By (2.6), $\varphi_\omega(t) = t - \omega^2/24\, t^3 + O(\omega^4)$, so

$$e^{x\varphi_\omega(t)} = e^{xt}\exp\left(-\frac{\omega^2}{24} x\, t^3 + O(\omega^4)\right) = e^{xt}\left(1 - \frac{\omega^2}{24} x\, t^3\right) + O(\omega^4).$$

Multiplying by $a(t)$ and using (2.2),

$$\sum_{n\ge 0} s_n^{[\omega]}(x)\frac{t^n}{n!} = \sum_{n\ge 0} A_n(x)\frac{t^n}{n!} - \frac{\omega^2}{24}\, x\, t^3 \sum_{n\ge 0} A_n(x)\frac{t^n}{n!} + O(\omega^4).$$

Multiplication by t^3 in the exponential normalisation sends $\sum_n c_n\, t^n/n!$ to $\sum_n \frac{n!}{(n-3)!} c_{n-3} t^n/n!$, and $\frac{n!}{(n-3)!} = n(n-1)(n-2)$. Comparing coefficients of $t^n/n!$ gives (5.1). ▫

Verification. For $a \equiv 1$ we have $A_n = x^n$, and (5.1) at $n = 3$ gives $x^3 - \omega^2/24 \cdot 6 \cdot x \cdot 1 = x^3 - \omega^2/4\, x$, in exact agreement with $x^{[3,\omega]}$ in (2.8). At $n = 4$ it gives $x^4 - \omega^2/24 \cdot 24 \cdot x \cdot x = x^4 - \omega^2 x^2$, agreeing with $x^{[4,\omega]} = x^2(x^2 - \omega^2)$.

Corollary 5.2 (Second-order connection coefficients). Write $a'(t)/a(t) = \sum_{j\ge 0} \lambda_j t^j$. Then, modulo $O(\omega^4)$,

$$c_{n,m}(\omega) = \delta_{n,m} + \frac{\omega^2}{24}\cdot\frac{n!}{m!}\left(\lambda_{n-m-3} - m\,\delta_{n,m+2}\right),$$

with $\lambda_j := 0$ for $j < 0$. In particular $c_{m+2,m} = -\frac{\omega^2}{24} m(m+1)(m+2) + O(\omega^4)$, and for $a \equiv 1$ all other subdiagonal entries vanish to this order, consistently with Corollary 4.6.

Proof. Expand (4.3) to first order in ω^2. From (2.6), $a(\varphi_\omega(t)) = a(t) - \omega^2/24\, t^3 a'(t) + O(\omega^4)$, so

$$\frac{a(t)}{a(\varphi_\omega(t))} = 1 + \frac{\omega^2}{24} t^3 \frac{a'(t)}{a(t)} + O(\omega^4),$$

while $\varphi_\omega(t)^m = t^m - m\omega^2/24\, t^{m+2} + O(\omega^4)$. Multiplying and extracting the coefficient of t^n gives $\delta_{n,m} + \omega^2/24\left(\lambda_{n-m-3} - m\delta_{n,m+2}\right)$ before the factor $n!/m!$. ▫

Remark 5.3. Expansion (5.1) is uniform over the Appell class: the correction depends on the sequence only through A_{n-3}, and its numerical coefficient $-n(n-1)(n-2)/24$ is universal. This is a practical statement — for small ω it permits evaluation of a central ω-Sheffer family from tabulated Appell values with a controlled error — and it is a structural one, since it identifies $x\,A_{n-3}$ as the unique first-order direction in which the deformation moves.

6. Determinantal Inversion

Determinantal representations of Appell and Sheffer families have been developed systematically in [12, 15]; the form below specialises that approach to the present pair.

Theorem 6.1. For $n \geq 0$,

$$A_n(x) = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & s_0^{[\omega]}(x) \\ c_{1,0} & 1 & 0 & \cdots & 0 & s_1^{[\omega]}(x) \\ c_{2,0} & c_{2,1} & 1 & \cdots & 0 & s_2^{[\omega]}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1,0} & c_{n-1,1} & c_{n-1,2} & \cdots & 1 & s_{n-1}^{[\omega]}(x) \\ c_{n,0} & c_{n,1} & c_{n,2} & \cdots & c_{n,n-1} & s_n^{[\omega]}(x) \end{vmatrix}.$$

Proof. Relations (4.1) for $0 \leq k \leq n$ form a linear system in $A_0, \dots, A_n$ whose coefficient matrix is lower triangular with unit diagonal by Corollary 4.2, hence of determinant 1. Cramer's rule gives $A_n = \det(M_n)$, where M_n replaces the last column $(0, \dots, 0,1)^T$ by $\left(s_0^{[\omega]}, \dots, s_n^{[\omega]}\right)^T$. ▫

The unit diagonal supplied by Corollary 4.2 is what removes the normalising factor that appears in determinantal formulas for general Sheffer families; this is a small but genuine simplification specific to the central case, where $\varphi'_{\omega}(0) = 1$ identically in ω.

Check at $n = 1$**.** Formula (6.1) gives $A_1 = s_1^{[\omega]} - c_{1,0}s_0^{[\omega]}$, which is the solution of $s_1^{[\omega]} = c_{1,0}A_0 + A_1$ with $A_0 = s_0^{[\omega]}$.

7. Examples

7.1 The ω-central factorials

Take $a \equiv 1$, so $A_n(x) = x^n$ and $s_n^{[\omega]} = x^{[n,\omega]}$. The first members are those listed in (2.8). By Corollary 4.6 the connection matrix has checkerboard support, and by Corollary 5.2 its first sub diagonal of interest is $c_{m+2,m} = -\omega^2/24\, m(m+1)(m+2) + O(\omega^4)$. At $m = 1$ this gives $c_{3,1} = -\omega^2/4 + O(\omega^4)$, and indeed

$$x^{[3,\omega]} = x^3 - \omega^2/4\, x = A_3(x) - \omega^2/4\, A_1(x),$$

so $c_{3,1} = -\omega^2/4$ exactly, $c_{3,2} = c_{3,0} = 0$. Similarly $x^{[4,\omega]} = x^4 - \omega^2 x^2$ gives $c_{4,2} = -\omega^2$, $c_{4,0} = c_{4,1} = c_{4,3} = 0$ — the checkerboard pattern in full.

Verification of the lowering relation: At $n = 3$,

$$\delta_\omega x^{[3,\omega]} = [(x + \omega/2)^3 - \omega^2/4\,(x + \omega/2)] - [(x - \omega/2)^3 - \omega^2/4\,(x - \omega/2)] = 3\omega x^2 = 3\omega\, x^{[2,\omega]},$$

in agreement with (2.10).

7.2 The ω-central Bernoulli polynomials

Take $a(t) = t/(e^t - 1)$, so $\{A_n\}$ is the Bernoulli sequence $B_n(x)$ [6, 7]. Definition 2.5 gives the ω**-central Bernoulli polynomials**

$$B_n^{[\omega]}(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}\, x^{[k,\omega]},$$

satisfying, by (2.10),

$$B_n^{[\omega]}(x+\omega/2)-B_n^{[\omega]}(x-\omega/2)=n\,\omega\,B_{n-1}^{[\omega]}(x).$$

The first members are

$$B_0^{[\omega]}=1,\qquad B_1^{[\omega]}=x-1/2\,,\qquad B_2^{[\omega]}=x^2-x+1/6,$$

$$B_3^{[\omega]}=x^3-3/2\,x^2+1/2\,x-\omega^2/4\,x.$$

Observe that $B_n^{[\omega]}=B_n$ for $n\le 2$, exactly as Theorem 5.1 predicts, the correction first appearing at $n=3$.

Connection coefficients. Here $a'(t)/a(t)=1/t-e^t/e^t-1$, whose expansion begins $\lambda_0=-1/2$. Corollary 5.2 gives

$$c_{3,0}=\frac{\omega^2}{24}\cdot 3!\cdot\lambda_0=-\frac{\omega^2}{8},\qquad c_{3,1}=\frac{\omega^2}{24}\cdot\frac{3!}{1!}\cdot(-1)=-\frac{\omega^2}{4},\qquad c_{3,2}=0,\qquad c_{3,3}=1,$$

to order ω^2. Substituting into (4.1),

$$c_{3,0}B_0+c_{3,1}B_1+B_3=-\omega^2/8-\omega^2/4\,(x-1/2)+(x^3-3/2\,x^2+1/2\,x)=x^3-3/2\,x^2+1/2\,x-\omega^2/4\,x,$$

which is exactly $B_3^{[\omega]}$ as displayed above, the constant terms $-\omega^2/8$ and $+\omega^2/8$ cancelling. Note that the correction is carried entirely by A_1 and A_0 even though the polynomial correction $-\omega^2/4\,x$ has no constant term: expanding $x=B_1(x)+1/2$ redistributes it. Note that $c_{3,0}\neq 0$: the checkerboard pattern of Corollary 4.6 fails here, as it must, since $a(t)/a\big(\varphi_\omega(t)\big)$ is not even.

Verification of (7.2) at $n=3$**.** With $B_3^{[\omega]}$ as above,

$$B_3^{[\omega]}(x+\omega/2)-B_3^{[\omega]}(x-\omega/2)=3\omega x^2-3\omega x+\omega/2=3\omega(x^2-x+1/6)=3\omega B_2^{[\omega]}(x),$$

as required. The cancellation of the ω^3 terms between $(x\pm\omega/2)^3$ and the correction $-\omega^2/4\,(x\pm\omega/2)$ is exactly what the ω^2-correction in $B_3^{[\omega]}$ is there to produce, and provides a stringent check on (7.1).

Numerical table. Values of $B_n^{[\omega]}(x)$ at $\omega=1$:

x	$B_0^{[1]}$	$B_1^{[1]}$	$B_2^{[1]}$	$B_3^{[1]}$
0	1	$-1/2$	$1/6$	0
1	1	$1/2$	$1/6$	$-1/4$
2	1	$3/2$	$13/6$	$5/2$

8. Concluding Remarks

The relation between the Appell class and the central ω-Sheffer class is realised by a single operator T_ω, determined by the step alone, conjugating the derivative into the normalised central difference. The connection problem between the two classes is governed by the conjugation $C(\omega)=\alpha\beta_\omega\alpha^{-1}$, and the parity of the associated delta series makes $C(\omega)$ an even polynomial deformation of the identity — a feature absent from the forward- and backward-difference theories. The resulting perturbative expansion (5.1) holds uniformly across the Appell class.

Several directions remain. The higher-order terms of (5.1) are governed by the coefficients of $\sinh^{-1}$ and should admit a closed form in terms of central factorial numbers; only the leading correction has been computed here. The similarity class of β_ω, and hence of every connection matrix arising from the same step, has not been determined, and its Jordan structure would settle Corollary 4.3 completely. Finally, the construction extends verbatim to any delta series that is odd in the indeterminate and even in the step, and it would be of interest to characterise which discrete operators [9, 10] arise this way. The determinantal and monomiality machinery of [11, 12, 14, 15] applies to any such class without modification.

10. References

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