

Total Outer- Connected Dominating Sets and Total Outer-Connected Domination Polynomials of Cycle Graph C_n

K. Lal Gipson¹, Gracelin Goldy J V²

¹Assistant Professor, ² Research Scholar, (Reg. No: 23213162092005) E-mail: lalgipson@yahoo.com

^{1,2} Department of Mathematics & Research Centre, Scott Christian College (Autonomous), Nagercoil – 629 003, Kanyakumari District, Tamil Nadu, India. gracelingoldy99@gmail.com²
(Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli -627 012, Tamil Nadu, India)

Abstract:

Let $G = (V, E)$ be a simple graph of order n . A dominating set D is a total outer- connected dominating set of G if D is total dominating and the induced subgraph $G[V(G) - D]$ is a connected graph. Let $\tilde{D}_{tc}(C_n, i)$ be the family of total outer-connected dominating sets of a cycle graph C_n with cardinality i and let $\tilde{D}_{tc}(C_n, i) = |\tilde{D}_{tc}(C_n, i)|$. In this paper we construct $\tilde{D}_{tc}(C_n, i)$ and obtain exact formula and recursive formulae for $\tilde{d}_{tc}(C_n, i)$. Using these formulae, we construct the polynomial $\tilde{D}_{tc}(C_n, x) = \sum_{i=\tilde{\gamma}_{tc}(C_n)}^{|V(G)|} \tilde{d}_{tc}(C_n, i)x^i$, which we call the total outer- connected domination polynomial and obtain some properties of this polynomial.

Keywords: Total outer- connected dominating set, total outer- connected domination number, total outer- connected domination polynomial.

Mathematics Subject Classification: 05C69, 05C31, 05

1. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ of G are denoted by n and m respectively. For any vertex $v \in V(G)$, the open neighbourhood of v is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the closed neighbourhood of v is the set $N_G[v] = N(v) \cup v$. For a set $S \subseteq V$, the open neighbourhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighbourhood of S is $N[S] = N(S) \cup S$.

A dominating set of G is a set $D \subseteq V(G)$ such that $N_G[v] \cap D \neq \emptyset$, for all $v \in V(G)$. The domination number of G is the minimum cardinality of a dominating set of G and it is denoted by $\gamma(G)$. Similarly, a total dominating set of G is a set $D \subseteq V(G)$ such that for each $v \in V(G)$, $N_G(v) \cap D \neq \emptyset$. The total domination number $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set of G .

Around 1960, domination was first explored mathematically. In the late 1950s and early 1960s, graph domination was further investigated when Claude Berge established the 'Coefficient of external stability' in 1958, which is commonly referred to as a graph's domination number. The terms 'Dominating Sets and Domination number' was first used by Oystein Ore in 1962. Cockayne and Hedetniemi were initially employed $\gamma(G)$ to determine the domination number of graph G . Dominating sets and domination polynomials of graphs were initially proposed by Saeid Alikhani and Yee hock Peng in 2009. The concept total dominating set was first introduced by Cockayne, Dawes and Hedetniemi in 1980. The concept of connected domination was first introduced by Sampath Kumar and Walikar in 1979. Joanna Cyman introduced the introduced the concept total outer- connected domination [7] and further explored in [8] together with Joanna Raczek. B. S. panda and Arti Pandey contributed NP Completeness in total outer connected domination [19]. Some bounds and generalization of total outer- connected domination was obtained by Nader Jafari Rad and Lut Volkmann [17]. In [9] Farshad Kazemnejad Behnaz Pahlavsay, Elisa palezzato and Michel Torielli obtained some results in total outer- connected domination number of middle graphs.

The total outer- connected domination parameter combines the coverage requirement of total domination with the connectivity condition imposed on the vertices outside the dominating set. This parameter is particularly relevant in the modelling of reliable communication and monitoring networks, where every vertex must be monitored while the unmonitored vertices maintain connectivity.

In recent years, total outer- connected domination has attracted considerable attention due to its rich mathematical properties and its connections with other domination parameters. The determination of total outer- connected domination numbers for various graph families, together with the establishment of sharp bounds and the investigation of associated domination polynomials, continues to be an active area of research. In this paper, we investigate the total outer- connected dominating sets of cycle graph C_n and derive the corresponding results of total outer- connected domination polynomial of cycle graph C_n .

Definition 1.1:

A cycle is a closed trail in which the vertices are all distinct. C_n denotes a cycle on n vertices.

Definition 1.2:

A set $S \subseteq V$ is a dominating set of G , if every vertex in $V - S$ is adjacent to at least one vertex in S . The domination number $\gamma(G)$ of G is the minimum cardinality of dominating set.

Definition 1.3: [7]

Let G be a simple connected graph. A set $D \subseteq V(G)$ is a total outer- connected dominating set of G if D is total dominating and the induced subgraph $G[V(G) - D]$ is a connected graph. The total outer- connected domination number of G , denoted by $\tilde{\gamma}_{tc}(G)$, is the minimum cardinality of total outer- connected dominating set of G .

Definition 1.4:

Let G be a simple connected graph. Let $\tilde{D}_{tc}(G, i)$ denote a family of all outer- connected dominating sets of G with cardinality i and let $|\tilde{D}_{tc}(G, i)|$. Then the total outer- connected domination polynomial $\tilde{D}_{tc}(G, x)$ of G is defined as $\tilde{D}_{tc}(G, x) = \sum_{i=\tilde{\gamma}_{tc}(G)}^{|V(G)|} |\tilde{D}_{tc}(G, i)| x^i$, where $\tilde{\gamma}_{tc}(G)$ is the total outer- connected domination number of G .

2. TOTAL OUTER- CONNECTED DOMINATING SETS OF CYCLE C_n .

The graph C_n called the Cycle graph of order n , consists of a sequence of n distinct vertices $u_1, u_2, \dots, u_n$ in which each vertex u_i is adjacent to u_{i+1} for $1 \leq i \leq n - 1$ and u_n is adjacent to u_1 there by forming a closed cycle. We shall investigate dominating sets of cycle graph. We need the following lemmas to prove our main results in this section.

Lemma 2.1:

Consider a graph C_n of order n . Thus, total outer- connected domination number

$$\gamma(C_n) = \begin{cases} n, & \text{if } n = 1, 2 \\ n - 1, & \text{if } n = 3 \\ n - 2, & \text{if } n \geq 4 \end{cases}$$

Lemma 2.2: Consider a graph C_n of order n and the collection of all total outer- connected dominating set of C_n is denoted as $\tilde{D}_{tc}(C_n, i)$. Thus,

- (i) $\tilde{D}_{tc}(C_n, i) = \emptyset$ if and only if $i < \tilde{\gamma}_{tc}(C_n)$ or $i > |V(C_n)|$.
- (ii) $\tilde{D}_{tc}(C_n, i) \neq \emptyset$ if and only if $\tilde{\gamma}_{tc}(C_n) \leq i \leq |V(C_n)|$.

Proof:

Consider $i = n - 2$, by the definition of total outer- connected domination number and lemma.2.1, there is a minimum of one total outer- connected dominating set in $\tilde{D}_{tc}(C_n, i)$ denote S . Evidently, $\tilde{D}_{tc}(C_n, i) \neq \emptyset$.

Assume, $i = n - 1$ and $i < n$, thus there exists a minimum of one vertex $u \in V - S$. Add u to S . Since, every super set of a total outer- connected dominating set is also a total outer- connected dominating set, $S \cup \{u\}$ is a total outer- connected dominating set having cardinality $i = n - 1$. Consequently, $\tilde{D}_{tc}(C_n, i) \neq \emptyset$.

In this way, we can prove that, $\tilde{D}_{tc}(C_n, i) \neq \emptyset$ is equivalent to $\tilde{\gamma}_{tc}(C_n) \leq i \leq |V(C_n)|$.

When, $i < \tilde{\gamma}_{tc}(C_n)$, thus there is no total outer- connected dominating set in $\tilde{D}_{tc}(C_n, i)$. Consequently, $\tilde{D}_{tc}(C_n, i) = \emptyset$ is equivalent to $i < \tilde{\gamma}_{tc}(C_n)$. Also, $\tilde{D}_{tc}(C_n, i) = \emptyset$ for all $i > |V(C_n)|$.

Thus, $\tilde{D}_{tc}(C_n, i) = \emptyset$ is equivalent to $i < \tilde{\gamma}_{tc}(C_n)$ or $i > |V(C_n)|$.

Lemma 2.3: Consider a graph C_n of order n . For all $n \geq 4$,

- (i) If $\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$ and $\tilde{D}_{tc}(C_{n-3}, i - 1) = \emptyset$ thus $\tilde{D}_{tc}(C_{n-2}, i - 1) = \emptyset$.
- (ii) If $\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$ and $\tilde{D}_{tc}(C_{n-3}, i - 1) = \emptyset$ then $\tilde{D}_{tc}(C_{n-2}, i - 1) \neq \emptyset$.
- (iii) If $\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$ and $\tilde{D}_{tc}(C_{n-3}, i - 1) \neq \emptyset$ thus $\tilde{D}_{tc}(C_{n-2}, i - 1) \neq \emptyset$.

Proof:

(i) As, $\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$ using lemma. 2.2 (ii), we obtain

$$\begin{aligned} n - 3 &\leq i - 1 \leq n - 1 \\ \Rightarrow i - 1 &> n - 2 \end{aligned} \tag{1}$$

As, $\tilde{D}_{tc}(C_{n-3}, i - 1) = \emptyset$, using lemma. 2.2 (i), we obtain

$$\begin{aligned} i - 1 &< n - 5 \text{ or } i - 1 > n - 3 \\ \Rightarrow i - 1 &< n - 4. \end{aligned} \tag{2}$$

Using equations (1) and (2), $i - 1 > n - 2$ or $i - 1 < n - 4$, based on lemma. 2.2 (i), we obtain

$\tilde{D}_{tc}(C_{n-2}, i - 1) = \emptyset$.

(ii) As, $\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$ using lemma. 4.2.2 (ii), we obtain

$$n - 3 \leq i - 1 \leq n - 1$$

$$\Rightarrow n - 4 \leq i - 1 \leq n - 2.$$

As, $\tilde{D}_{tc}(C_{n-3}, i - 1) = \emptyset$, using lemma. 4.2.2 (ii), we obtain

$i - 1 < n - 5$ or $i - 1 > n - 3$ (skip this case)

Based on lemma. 2.2 (ii), we obtain $\tilde{D}_{tc}(C_{n-2}, i - 1) = \emptyset$.

(iii) As, $\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$ using lemma. 2.2 (ii), we obtain

$n - 3 \leq i - 1 \leq n - 1$ (skip this case)

As, $\tilde{D}_{tc}(C_{n-3}, i - 1) \neq \emptyset$, using lemma. 2.2 (ii), we obtain

$$\begin{aligned} n - 5 \leq i - 1 \leq n - 3 \\ \Rightarrow n - 4 \leq i - 1 \leq n - 2. \end{aligned}$$

Based on lemma. 2.2 (ii), we obtain $\tilde{D}_{tc}(C_{n-2}, i - 1) \neq \emptyset$.

Lemma 2.4: Consider a graph C_n of order n . Assume that, $\tilde{D}_{tc}(C_n, i) \neq \emptyset$, thus

(i) $\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$, $\tilde{D}_{tc}(C_{n-2}, i - 1) \neq \emptyset$ and $\tilde{D}_{tc}(C_{n-3}, i - 1) \neq \emptyset$ equivalently, $i = n - 2$.

(ii) $\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$, $\tilde{D}_{tc}(C_{n-2}, i - 1) \neq \emptyset$ and $\tilde{D}_{tc}(C_{n-3}, i - 1) = \emptyset$ equivalently, $i = n - 1$.

(iii) $\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$, $\tilde{D}_{tc}(C_{n-2}, i - 1) = \emptyset$ and $\tilde{D}_{tc}(C_{n-3}, i - 1) = \emptyset$ equivalently, $i = n$.

Proof:

(i) ($\Rightarrow$) As, $\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$ using lemma 2.2(ii), we obtain

$$\begin{aligned} (n - 1) - 2 \leq i - 1 \leq n - 1 \\ \Rightarrow n - 3 \leq i - 1 \leq n - 1 \\ \Rightarrow n - 2 \leq i \leq n \\ \Rightarrow i \geq n - 2 \end{aligned} \quad (3)$$

($\Leftarrow$) As, $\tilde{D}_{tc}(C_{n-3}, i - 1) \neq \emptyset$ using lemma 2.2(ii), we obtain

$$\begin{aligned} (n - 3) - 2 \leq i - 1 \leq n - 3 \\ \Rightarrow n - 5 \leq i - 1 \leq n - 3 \\ \Rightarrow n - 4 \leq i \leq n - 2 \\ \Rightarrow i \leq n - 2 \end{aligned} \quad (4)$$

Using equations (3) and (4), we obtain $i = n - 2$.

($\Leftarrow$) Assume that $i = n - 2$, means $i - 1 \leq n - 3$, using lemma. 2.2 (ii), $\tilde{D}_{tc}(C_{n-3}, i - 1) \neq \emptyset$. Analogously,

$\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$ and $\tilde{D}_{tc}(C_{n-2}, i - 1) \neq \emptyset$. Thus,

$$\tilde{D}_{tc}(C_{n-1}, i - 1) = \tilde{D}_{tc}(C_{n-1}, n - 3) \neq \emptyset$$

$$\tilde{D}_{tc}(C_{n-2}, i - 1) = \tilde{D}_{tc}(C_{n-2}, n - 3) \neq \emptyset$$

$$\tilde{D}_{tc}(C_{n-3}, i - 1) = \tilde{D}_{tc}(C_{n-3}, n - 3) \neq \emptyset.$$

(ii) ($\Rightarrow$) As, $\tilde{D}_{tc}(C_{n-2}, i - 1) \neq \emptyset$ using lemma 2.2(ii), we obtain

$$\begin{aligned} n - 4 \leq i - 1 \leq n - 2 \\ \Rightarrow n - 3 \leq i \leq n - 1 \\ \Rightarrow i \leq n - 1 \end{aligned} \quad (5)$$

As, $\tilde{D}_{tc}(C_{n-3}, i - 1) = \emptyset$ using lemma 2.2(i), we obtain

$$\begin{aligned} i - 1 < n - 5 \text{ (skip this case) or } i - 1 > n - 3 \\ \Rightarrow i > n - 2 \\ \Rightarrow i \geq n - 1 \end{aligned} \quad (6)$$

Using equations (5) and (6), we obtain $i = n - 1$.

($\Leftarrow$) Assume, $i = n - 1$, means $i - 1 \geq n - 2$, we obtain $\tilde{D}_{tc}(C_{n-2}, i - 1) \neq \emptyset$ using lemma. 2.2(ii).

Analogously, $\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$. As, $i - 1 \leq n - 3$, using lemma. 4.2.2(i), $\tilde{D}_{tc}(C_{n-3}, i - 1) = \emptyset$. Thus,

$$\tilde{D}_{tc}(C_{n-1}, i - 1) = \tilde{D}_{tc}(C_{n-1}, n - 2) \neq \emptyset$$

$$\tilde{D}_{tc}(C_{n-2}, i - 1) = \tilde{D}_{tc}(C_{n-2}, n - 2) \neq \emptyset$$

$$\tilde{D}_{tc}(C_{n-3}, i - 1) = \tilde{D}_{tc}(C_{n-3}, n - 2) = \emptyset.$$

(iii) ($\Rightarrow$) As, $\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$ using lemma 2.2 (ii), we obtain

$$\begin{aligned} n - 3 \leq i - 1 \leq n - 1 \\ \Rightarrow n - 2 \leq i \leq n \\ \Rightarrow i \leq n \end{aligned} \quad (7)$$

As, $\tilde{D}_{tc}(C_{n-2}, i - 1) = \emptyset$ using lemma 2.2 (i), we obtain

$$\begin{aligned} i - 1 < n - 4 \text{ (skip this case) or } i - 1 > n - 2 \\ \Rightarrow i > n - 1 \\ \Rightarrow i \geq n \end{aligned} \quad (8)$$

Using equations (7) and (8), we obtain $i = n$.

($\Leftarrow$) Assume, $i = n$, means $i - 1 > n - 2$, we obtain $\tilde{D}_{tc}(C_{n-2}, i - 1) = \emptyset$ using lemma. 4.2.2 (i).

Analogously, $\tilde{D}_{tc}(C_{n-3}, i - 1) = \emptyset$. As, $i - 1 = n - 1$, using lemma. 2.2 (ii), $\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$. Thus,

$$\tilde{D}_{tc}(C_{n-1}, i - 1) = \tilde{D}_{tc}(C_{n-1}, n - 1) \neq \emptyset$$

$$\tilde{D}_{tc}(C_{n-1}, i - 1) = \tilde{D}_{tc}(C_{n-2}, n - 1) = \emptyset$$

$$\tilde{D}_{tc}(C_{n-1}, i - 1) = \tilde{D}_{tc}(C_{n-3}, n - 1) = \emptyset.$$

The theorem stated below characterizes the families of total outer-connected dominating sets of C_n .

Theorem 2.5: For every $n \in N$ and $i \geq n - 2$,

(i) When, $\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$, $\tilde{D}_{tc}(C_{n-2}, i - 1) = \emptyset$ and $\tilde{D}_{tc}(C_{n-3}, i - 1) = \emptyset$, thus $\tilde{D}_{tc}(C_n, i) = \{[n]\}$ if $i = n$.

(ii) When, $\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$, $\tilde{D}_{tc}(C_{n-2}, i - 1) \neq \emptyset$ and $\tilde{D}_{tc}(C_{n-3}, i - 1) = \emptyset$, thus

$$\tilde{D}_{tc}(C_n, i) = \left\{ \left\{ X_1 \cup \{u_n\} / X_1 \in \tilde{D}_{tc}(C_{n-1}, i - 1) \right\} \cup \left\{ X_2 \cup \{u_{n-1}\} / X_2 \in \tilde{D}_{tc}(C_{n-2}, i - 1) \right\} \right\}$$

if $i = n - 1$.

(iii) When, $\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$, $\tilde{D}_{tc}(C_{n-2}, i - 1) \neq \emptyset$ and $\tilde{D}_{tc}(C_{n-3}, i - 1) \neq \emptyset$, thus

$$\tilde{D}_{tc}(C_n, i) = \left\{ \left\{ X_1 \cup \begin{cases} \{u_{n-1}\}, & \text{if } X_1 \text{ terminates in } u_{n-2} \\ \{u_n\}, & \text{if } X_1 \text{ terminates in } u_{n-3} \text{ and } u_{n-1} \end{cases} / X_1 \in \tilde{D}_{tc}(C_{n-1}, i - 1) \right\} \cup \left\{ X_2 \cup \{u_{n-2}\} / X_2 \in \tilde{D}_{tc}(C_{n-3}, i - 1) \right\} \right\}$$

if $i = n - 2$.

Proof:

Consider a graph C_n of order n .

(i) As, $\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$, $\tilde{D}_{tc}(C_{n-2}, i - 1) = \emptyset$ and $\tilde{D}_{tc}(C_{n-3}, i - 1) = \emptyset$ using lemma. 2.4 (i), we obtain $i = n$. So, $\tilde{D}_{tc}(C_n, i) = \tilde{D}_{tc}(C_n, n) = \{[n]\}$.

(ii) As, $\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$, $\tilde{D}_{tc}(C_{n-2}, i - 1) \neq \emptyset$ and $\tilde{D}_{tc}(C_{n-3}, i - 1) = \emptyset$.

Case: 1:

Consider, X_1 be a total outer-connected dominating set of C_{n-1} having cardinality $i - 1$. The components of $\tilde{D}_{tc}(C_{n-1}, i - 1)$ terminates in u_{n-1} or u_{n-2} .

Sub Case: 1:

When $u_{n-1} \in X_1$ and $u_{n-2} \in X_1$, thus the components of $\tilde{D}_{tc}(C_{n-1}, i - 1)$ is a component of $\tilde{D}_{tc}(C_n, i)$ through the addition of u_n .

Case: 2:

Consider X_2 be a total outer-connected dominating set of C_{n-2} having cardinality $i - 1$. The components of $\tilde{D}_{tc}(C_n, i)$ terminates in u_{n-2} .

Sub Case:1:

When, $u_{n-2} \in X_2$, the components of $\tilde{D}_{tc}(C_{n-2}, i - 1)$ are a component of $\tilde{D}_{tc}(C_n, i)$ through the addition of u_{n-1} .

$$\text{Thus, } \left\{ \left\{ X_1 \cup \{u_n\} / X_1 \in \tilde{D}_{tc}(C_{n-1}, i - 1) \right\} \cup \left\{ X_2 \cup \{u_{n-1}\} / X_2 \in \tilde{D}_{tc}(C_{n-2}, i - 1) \right\} \right\} \in \tilde{D}_{tc}(C_n, i).$$

Likewise, assume that $Z \in \tilde{D}_{tc}(C_n, i)$, every component of $\tilde{D}_{tc}(C_n, i)$ terminates in u_{n-1} or u_n .

Case:1:

When $u_{n-1} \in Z$ and $u_n \notin Z$, thus a minimum of one vertex is numbered as u_{n-2} .

Sub Case:1:

Assume, $u_{n-2} \in Z$. Thus, $Z = X_2 \cup \{u_{n-1}\}$ for some, $X_2 \in \tilde{D}_{tc}(C_{n-2}, i - 1)$.

Case:2:

When $u_n \in Z$ and $u_{n-1} \notin Z$, thus a minimum of one vertex is numbered as u_{n-2} or u_{n-1} .

Sub Case:1:

Assume, $u_{n-2} \in Z$, $u_{n-3} \notin Z$. Thus, $Z = X_1 \cup \{u_n\}$ for some, $X_1 \in \tilde{D}_{tc}(C_{n-1}, i - 1)$.

Sub Case:2:

Assume, $u_{n-1} \in Z$, $u_{n-2} \notin Z$. Thus, $Z = X_1 \cup \{u_n\}$ for some, $X_1 \in \tilde{D}_{tc}(C_{n-1}, i - 1)$.

$$\text{Thus, } \tilde{D}_{tc}(C_n, i) \subseteq \left\{ \left\{ X_1 \cup \{u_n\} / X_1 \in \tilde{D}_{tc}(C_{n-1}, i - 1) \right\} \cup \left\{ X_2 \cup \{u_{n-1}\} / X_2 \in \tilde{D}_{tc}(C_{n-2}, i - 1) \right\} \right\}.$$

$$\text{Thus, it follows that, } \tilde{D}_{tc}(C_n, i) = \left\{ \left\{ X_1 \cup \{u_n\} / X_1 \in \tilde{D}_{tc}(C_{n-1}, i - 1) \right\} \cup \left\{ X_2 \cup \{u_{n-1}\} / X_2 \in \tilde{D}_{tc}(C_{n-2}, i - 1) \right\} \right\}.$$

(iii) As, $\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$, $\tilde{D}_{tc}(C_{n-2}, i - 1) \neq \emptyset$ and $\tilde{D}_{tc}(C_{n-3}, i - 1) \neq \emptyset$.

Case:1:

Consider X_1 be a total outer- connected dominating set of C_{n-1} having cardinality $i - 1$. The components of $\tilde{D}_{tc}(C_{n-1}, i - 1)$ terminates with u_{n-1} , u_{n-2} and u_{n-3} .

Sub Case:1:

When $u_{n-1} \in X_1$, $u_{n-2} \notin X_1$ and $u_{n-3} \notin X_1$ thus the components of $\tilde{D}_{tc}(C_{n-1}, i - 1)$ is a component of $\tilde{D}_{tc}(C_n, i)$ through the addition of u_n .

Sub Case:2:

When $u_{n-2} \in X_1$, $u_{n-1} \notin X_1$ and $u_{n-3} \notin X_1$ thus the components of $\tilde{D}_{tc}(C_{n-1}, i - 1)$ is a component of $\tilde{D}_{tc}(C_n, i)$ through the addition of u_{n-1} .

Case:2:

Consider, X_2 be a total outer- connected dominating set of C_{n-3} having cardinality $i - 1$. The components of $\tilde{D}_{tc}(C_{n-3}, i - 1)$ terminates with u_{n-3} .

Sub Case:1:

When $u_{n-3} \in X_2$, thus the components of $\tilde{D}_{tc}(C_{n-3}, i - 1)$ is a component of $\tilde{D}_{tc}(C_n, i)$ through the addition of u_{n-2} .

$$\text{Thus, } \left\{ \left\{ X_1 \cup \begin{cases} \{u_{n-1}\}, & \text{if } X_1 \text{ terminates in } u_{n-2} \\ \{u_n\}, & \text{if } X_1 \text{ terminates in } u_{n-3} \text{ and } u_{n-1} \end{cases} \right\} / X_1 \in \tilde{D}_{tc}(C_{n-1}, i - 1) \right\} \cup \left\{ \begin{matrix} X_2 \cup \{u_{n-2}\} / X_2 \in \tilde{D}_{tc}(C_{n-3}, i - 1) \end{matrix} \right\} \subseteq \tilde{D}_{tc}(C_n, i)$$

Likewise, assume that $Z \in \tilde{D}_{tc}(C_n, i)$, every component of $\tilde{D}_{tc}(C_n, i)$ terminates with u_{n-2} , u_{n-1} and u_n .

Case:1:

When $u_{n-2} \in Z$, $u_{n-1} \notin Z$ and $u_n \notin Z$, thus a minimum of one vertex is numbered as u_{n-3} .

Sub case:1:

Assume, $u_{n-3} \in Z$. Thus, $Z = X_2 \cup \{u_{n-2}\}$ for some, $X_2 \in \tilde{D}_{tc}(C_{n-3}, i - 1)$.

Case:2:

When $u_{n-1} \in Z$, $u_{n-2} \notin Z$ and $u_n \notin Z$, thus a minimum of one vertex is numbered as u_{n-2} .

Sub Case:1:

Assume, $u_{n-2} \in Z$. Thus, $Z = X_1 \cup \{u_{n-1}\}$ for some, $X_1 \in \tilde{D}_{tc}(C_{n-1}, i - 1)$.

Case:3:

When $u_n \in Z$, $u_{n-1} \notin Z$ and $u_{n-2} \notin Z$, thus a minimum of one vertex is numbered as u_{n-3} and u_{n-1} .

Sub Case:1:

Assume, $u_{n-3} \in Z$ and $u_{n-1} \notin Z$. Thus, $Z = X_1 \cup \{u_n\}$ for some, $X_1 \in \tilde{D}_{tc}(C_{n-1}, i - 1)$.

Sub Case:2:

Assume, $u_{n-1} \in Z$ and $u_{n-3} \notin Z$. Thus, $Z = X_1 \cup \{u_n\}$ for some, $X_1 \in \tilde{D}_{tc}(C_{n-1}, i - 1)$.

$$\text{Thus, } \tilde{D}_{tc}(C_n, i) \subseteq \left\{ \left\{ X_1 \cup \begin{cases} \{u_{n-1}\}, & \text{if } X_1 \text{ terminates in } u_{n-2} \\ \{u_n\}, & \text{if } X_1 \text{ terminates in } u_{n-3} \text{ and } u_{n-1} \end{cases} \right\} / X_1 \in \tilde{D}_{tc}(C_{n-1}, i - 1) \right\} \cup \left\{ \begin{matrix} X_2 \cup \{u_{n-2}\} / X_2 \in \tilde{D}_{tc}(C_{n-3}, i - 1) \end{matrix} \right\}$$

Thus, it follows that,

$$\tilde{D}_{tc}(C_n, i) = \left\{ \left\{ X_1 \cup \begin{cases} \{u_{n-1}\}, & \text{if } X_1 \text{ terminates in } u_{n-2} \\ \{u_n\}, & \text{if } X_1 \text{ terminates in } u_{n-3} \text{ and } u_{n-1} \end{cases} \right\} / X_1 \in \tilde{D}_{tc}(C_{n-1}, i - 1) \right\} \cup \left\{ \begin{matrix} X_2 \cup \{u_{n-2}\} / X_2 \in \tilde{D}_{tc}(C_{n-3}, i - 1) \end{matrix} \right\}$$

3. TOTAL OUTER- CONNECTED DOMINATION POLYNOMIAL OF CYCLE GRAPH

C_n .

In this section we introduce and investigate the total outer- connected domination polynomial of cycle graph C_n .

Theorem 3.1: For every $n \geq 5$,

$$\tilde{d}_{tc}(C_n, i) = \begin{cases} \tilde{d}_{tc}(C_n, i - 1) + 1 & \text{if } i = n - 1, n - 2 \\ \tilde{d}_{tc}(C_{n-1}, i - 1) \text{ or } 1 & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

Proof:

(i) When $i = n - 1$, $\tilde{D}_{tc}(C_{n-1}, i - 1) \neq \emptyset$, $\tilde{D}_{tc}(C_{n-2}, i - 1) \neq \emptyset$ and $\tilde{D}_{tc}(C_{n-3}, i - 1) = \emptyset$, using lemma. 2.5 (ii), we obtain

$$\tilde{D}_{tc}(C_n, i) = \left\{ \{X_1 \cup \{u_n\}/X_1 \in \tilde{D}_{tc}(C_{n-1}, i-1)\} \cup \{X_2 \cup \{u_{n-1}\}/X_2 \in \tilde{D}_{tc}(C_{n-2}, i-1)\} \right\}$$

$$\Rightarrow |\tilde{D}_{tc}(C_n, i)| = |\tilde{D}_{tc}(C_{n-1}, i-1)| + |\tilde{D}_{tc}(C_{n-2}, i-1)|.$$

$$\Rightarrow \tilde{d}_{tc}(C_n, i) = \tilde{d}_{tc}(C_{n-1}, i-1) + 1.$$

When $i = n-2$, $\tilde{D}_{tc}(C_{n-1}, i-1) \neq \emptyset$, $\tilde{D}_{tc}(C_{n-2}, i-1) \neq \emptyset$ and $\tilde{D}_{tc}(C_{n-3}, i-1) \neq \emptyset$, using lemma. 2.5 (iii), we obtain

$$\tilde{D}_{tc}(C_n, i) = \left\{ \left\{ X_1 \cup \begin{cases} \{u_{n-1}\}, & \text{if } X_1 \text{ terminates in } u_{n-2} \\ \{u_n\}, & \text{if } X_1 \text{ terminates in } u_{n-3} \text{ and } u_{n-1} \end{cases} / X_1 \in \tilde{D}_{tc}(C_{n-1}, i-1) \right\} \cup \left\{ X_2 \cup \{u_{n-2}\}/X_2 \in \tilde{D}_{tc}(C_{n-3}, i-1) \right\} \right\}$$

$$\Rightarrow |\tilde{D}_{tc}(C_n, i)| = |\tilde{D}_{tc}(C_{n-1}, i-1)| + |\tilde{D}_{tc}(C_{n-3}, i-1)|.$$

$$\Rightarrow \tilde{d}_{tc}(C_n, i) = \tilde{d}_{tc}(C_{n-1}, i-1) + 1$$

(ii) When $i = n$, $\tilde{D}_{tc}(C_{n-1}, i-1) \neq \emptyset$, $\tilde{D}_{tc}(C_{n-2}, i-1) \neq \emptyset$ and $\tilde{D}_{tc}(C_{n-3}, i-1) = \emptyset$, using lemma. 2.5 (i), we obtain

$$\tilde{D}_{tc}(C_n, i) = \{[n]\}$$

$$\Rightarrow |\tilde{D}_{tc}(C_n, i)| = |\tilde{D}_{tc}(C_{n-1}, i-1)|$$

$$\Rightarrow \tilde{d}_{tc}(C_n, i) = \tilde{d}_{tc}(C_{n-1}, i-1) \text{ or } \tilde{d}_{tc}(C_n, i) = 1$$

$$\text{Thus, } \tilde{d}_{tc}(C_n, i) = \begin{cases} \tilde{d}_{tc}(C_n, i-1) + 1 & \text{if } i = n-1, n-2 \\ \tilde{d}_{tc}(C_{n-1}, i-1) \text{ or } 1 & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

n \ i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
1	1														
2	0	1													
3	0	3	1												
4	0	4	4	1											
5	0	0	5	5	1										
6	0	0	0	6	6	1									
7	0	0	0	0	7	7	1								
8	0	0	0	0	0	8	8	1							
9	0	0	0	0	0	0	9	9	1						
10	0	0	0	0	0	0	0	10	10	1					
11	0	0	0	0	0	0	0	0	11	11	1				
12	0	0	0	0	0	0	0	0	0	12	12	1			
13	0	0	0	0	0	0	0	0	0	0	13	13	1		
14	0	0	0	0	0	0	0	0	0	0	0	14	14	1	
15	0	0	0	0	0	0	0	0	0	0	0	0	15	15	1

Table 1: $\tilde{d}_{tc}(C_n, i)$, the number of total outer-connected dominating sets of C_n having cardinality i .

Theorem 3.2: For all $n \geq 4$, $\tilde{D}_{tc}(C_n, x) = x\tilde{D}_{tc}(C_{n-1}, x) + x^{n-2} + x^{n-1}$ with initial values $\tilde{D}_{tc}(C_1, x) = x$, $\tilde{D}_{tc}(C_2, x) = x^2$, $\tilde{D}_{tc}(C_3, x) = 3x^2 + x^3$.

Proof:

Using definition. 3.1, we obtain

$$\begin{aligned} \tilde{D}_{tc}(C_n, x) &= \sum_{i=n-2}^n \tilde{d}_{tc}(C_n, i)x^i \\ &= \tilde{d}_{tc}(C_n, n-2)x^{n-2} + \tilde{d}_{tc}(C_n, n-1)x^{n-1} + \tilde{d}_{tc}(C_n, n)x^n \\ &= [\tilde{d}_{tc}(C_{n-1}, n-3) + 1]x^{n-2} + [\tilde{d}_{tc}(C_{n-1}, n-2) + 1]x^{n-1} + [\tilde{d}_{tc}(C_{n-1}, n-1)]x^n \\ &= x[\tilde{d}_{tc}(C_{n-1}, n-3)x^{n-3} + \tilde{d}_{tc}(C_{n-1}, n-2)x^{n-2}] + \tilde{d}_{tc}(C_{n-1}, n-1)x^{n-1} + x^{n-2} + x^{n-1} \end{aligned}$$

$$= x \sum_{i=n-2}^n \tilde{d}_{tc}(C_{n-1}, i-1) x^{i-1} + x^{n-2} + x^{n-1}$$

$$= x \tilde{D}_{tc}(C_{n-1}, x) + x^{n-2} + x^{n-1}, n \geq 4.$$

Theorem 3.3: The total outer- connected domination polynomial of cycle graph C_n is $x^n + n x^{n-2} \{x + 1\}$, for all $n \geq 4$, with initial values $\tilde{D}_{tc}(C_1, x) = x$, $\tilde{D}_{tc}(C_2, x) = x^2$, $\tilde{D}_{tc}(C_3, x) = 3x^2 + x^3$.

Proof:

Consider a graph C_n of order n . Consider, $V(C_n) = \{u_1, u_2, \dots, u_n\}$, the vertices of C_n . There is no total outer- connected dominating sets of cardinalities less than $n - 3$.

Case:1:

There are n total outer- connected dominating sets of cardinality $n - 2$. This is the minimum total outer- connected dominating set. This implies that, $\tilde{d}_{tc}(C_n, n - 2) = n$.

Case:2:

There is n total outer- connected dominating sets of cardinality $n - 1$. This implies that, $\tilde{d}_{tc}(C_n, n - 1) = n$.

Case:3:

There are only one total outer- connected dominating sets of cardinality n . This implies that, $\tilde{d}_{tc}(C_n, n) = 1$.

Thus, the minimum cardinality of total outer- connected dominating set of cycle C_n is $\tilde{\gamma}_{tc}(C_n) = n - 2$. Consequently,

$$\tilde{D}_{tc}(C_n, x) = \sum_{i=n-2}^n \tilde{d}_{tc}(C_n, i) x^i$$

$$= n x^{n-2} + n x^{n-1} + x^n$$

$$= x^n + n x^{n-2} \{x + 1\}, n \geq 4.$$

Remark 3.4. The co-efficient sum of the total outer- connected domination polynomial of cycle graph C_n is $2n + 1$, for all $n \geq 4$.

Theorem 3.5. The co-efficient of $\tilde{D}_{tc}(C_n, x)$, satisfy the following properties.

(i) $\tilde{d}_{tc}(C_n, n) = 1$, for all $n \in N$.

(ii) $\tilde{d}_{tc}(C_{n+1}, n) = n + 1$, for all $n \geq 2$.

(iii) $\tilde{D}_{tc}(C_{n+2}, n) = n + 2$, for all $n \geq 2$.

(iv) $\tilde{d}_{tc}(C_{2n+1}, 2n) = 2n + 1$, for all $n \in N$.

(v) $\tilde{d}_{tc}(C_{2n+2}, 2n + 1) = 2n + 2$, for all $n \in N$.

(vi) $\tilde{d}_{tc}(C_{ni}, ni - 2) = \tilde{d}_{tc}(C_{ni}, ni - 1) = ni$, for all $n \in N$ and $i \geq 4$.

(vii) $\tilde{d}_{tc}(C_{ni+1}, ni - 1) = ni + 1$, for all $n \in N$ and $i \geq 3$.

(viii) $\sum_{n=i}^{i+2} \tilde{d}_{tc}(C_n, i) = \sum_{n=i-1}^{i+1} \tilde{d}_{tc}(C_n, i - 1) + \sum_{n=1}^2 \tilde{d}_{tc}(C_n, n)$, for all $i \geq 3$.

(ix) $S_n = \sum_{i=n-2}^n \tilde{d}_{tc}(C_n, i)$ thus, for all $n \geq 5$, $S_n = S_{n-1} + S_1 + S_2$, having initial values $S_1 = 1, S_2 = 1, S_3 = 4, S_4 = 9$.

Proof:

Consider a graph C_n of order n . Consider, $V(C_n) = \{u_1, u_2, \dots, u_n\}$.

(i) As, $\tilde{d}_{tc}(C_n, n) = \{[n]\}$, consequently, $\tilde{d}_{tc}(C_n, n) = 1$, for all $n \in N$.

(ii) To demonstrate, $\tilde{d}_{tc}(C_{n+1}, n) = n + 1$, for all $n \geq 2$. The proof proceeds by mathematical induction on n . For, $n = 2$,

From Table: $\tilde{d}_{tc}(C_{2+1}, 2) = \tilde{d}_{tc}(C_3, 2) = 3$.

From Formula: $n + 1 = 2 + 1 = 3$.

It follows that, for $n = 2$, the obtained result is valid. we further establish the validity of the result for n . Suppose, inductively, that the result is valid for every $m < n, m \in N$. Using inductive assumption and theorem. 3.1,

$$\tilde{d}_{tc}(C_{n+1}, n) = \tilde{d}_{tc}(C_{(n-1)+1}, n - 1) + 1$$

$$= n - 1 + 1 + 1$$

$$= n + 1.$$

Consequently, $\tilde{d}_{tc}(C_{n+1}, n) = n + 1$, for all $n \geq 2$.

(iii) To demonstrate $\tilde{d}_{tc}(C_{n+2}, n) = n + 2$, for all $n \geq 2$. The proof proceeds by mathematical induction on n . For, $n = 2$,

From Table: $\tilde{d}_{tc}(C_{2+2}, 2) = \tilde{d}_{tc}(C_4, 2) = 4$.

From Formula: $n + 2 = 2 + 2 = 4$.

It follows that for $n = 2$, the obtained result is valid. we further establish the validity of the result for n . Suppose, inductively, that the result is valid for every $m < n, m \in N$. Using inductive assumption and theorem. 3.1,

$$\begin{aligned}\tilde{d}_{tc}(C_{n+2}, n) &= \tilde{d}_{tc}(C_{(n-1)+2}, n-1) + 1 \\ &= n - 1 + 2 + 1\end{aligned}$$

$$= n + 1.$$

Consequently, $\tilde{d}_{tc}(C_{n+2}, n) = n + 2$, for all $n \geq 2$.

(iv) To demonstrate, $\tilde{d}_{tc}(C_{2n+1}, 2n) = 2n + 1$, for all $n \in N$. The proof proceeds by mathematical induction on n . For, $n = 1$,

From Table: $\tilde{d}_{tc}(C_{2(1)+1}, 2(1)) = \tilde{d}_{tc}(C_3, 2) = 3$.

From Formula: $2n + 1 = 2(1) + 1 = 2 + 1 = 3$.

It follows that for $n = 1$, the obtained result is valid. we further establish the validity of the result for n . Suppose, inductively, that the result is valid for every $m < n, m \in N$. Using inductive assumption and theorem. 3.1,

$$\begin{aligned}\tilde{d}_{tc}(C_{2n+1}, 2n) &= \tilde{d}_{tc}(C_{(2n-1)+1}, 2n-1) + 1 \\ &= 2n - 1 + 1 + 1 \\ &= 2n + 1.\end{aligned}$$

Consequently, $\tilde{d}_{tc}(C_{2n+1}, 2n) = 2n + 1$, for all $n \in N$.

(v) To demonstrate, $\tilde{d}_{tc}(C_{2n+2}, 2n+1) = 2n + 2$, for all $n \in N$. The proof proceeds by mathematical induction on n . For, $n = 1$,

From Table: $\tilde{d}_{tc}(C_{2(1)+2}, 2(1)+1) = \tilde{d}_{tc}(C_{2+2}, 2+1) = \tilde{d}_{tc}(C_4, 3) = 4$.

From Formula: $2(1) + 2 = 2 + 2 = 4$.

It follows that for $n = 1$, the obtained result is valid. we further establish the validity of the result for n . Suppose, inductively, that the result is valid for every $m < n, m \in N$. Using inductive assumption and theorem. 3.1,

$$\begin{aligned}\tilde{d}_{tc}(C_{2n+2}, 2n+1) &= \tilde{d}_{tc}(C_{(2n-1)+2}, (2n-1)+1) + 1 \\ &= 2n - 1 + 2 + 1 \\ &= 2n + 2.\end{aligned}$$

Consequently, $\tilde{d}_{tc}(C_{2n+2}, 2n+1) = 2n + 2$, for all $n \in N$.

(vi) To demonstrate, $\tilde{d}_{tc}(C_{ni}, ni-2) = ni$, for all $n \in N$ and $i \geq 4$. The proof proceeds by mathematical induction on n . For, $n = 1$ and $i = 4$,

From Table: $\tilde{d}_{tc}(C_{4(1)}, 4(1)-2) = \tilde{d}_{tc}(C_4, 4-2) = \tilde{d}_{tc}(C_4, 2) = 3$.

From Formula: $ni = 4(1) = 4$.

It follows that for $n = 1$ and $i = 4$, the obtained result is valid. We further establish the validity of the result for n . Suppose, inductively, that the result is valid for every $m < n, m \in N$ and $j < i, j \in N$. Using inductive assumption and theorem. 3.1,

$$\begin{aligned}\tilde{d}_{tc}(C_{ni}, ni-2) &= \tilde{d}_{tc}(C_{ni-1}, (ni-2)-1) + 1 \\ &= \tilde{d}_{tc}(C_{ni-1}, (ni-1)-2) + 1 \\ &= ni - 1 + 1 \\ &= ni.\end{aligned}$$

Consequently, $\tilde{d}_{tc}(C_{ni}, ni-2) = ni$, for all $n \in N$ and $i \geq 4$.

Likewise, it can be proved that, $\tilde{d}_{tc}(C_{ni}, ni-1) = ni$, for all $n \in N$ and $i \geq 4$.

It follows that, $\tilde{d}_{tc}(C_{ni}, ni-2) = ni = \tilde{d}_{tc}(C_{ni}, ni-1) = ni$, for all $n \in N$ and $i \geq 4$.

(vii) To demonstrate, $\tilde{d}_{tc}(C_{ni+1}, ni-1) = ni + 1$, for all $n \in N$ and $i \geq 3$. The proof proceeds by mathematical induction on n . For, $n = 1$ and $i = 3$,

From Table: $\tilde{d}_{tc}(C_{3(1)+1}, 3(1)-1) = \tilde{d}_{tc}(C_{3+1}, 3-1) = \tilde{d}_{tc}(C_4, 2) = 4$.

From Formula: $ni + 1 = 3(1) + 1 = 4$.

It follows that for $n = 1$ and $i = 3$, the obtained result is valid. We further establish the validity of the result for n . Suppose, inductively, that the result is valid for every $m < n, m \in N$ and $j < i, j \in N$. Using inductive assumption and theorem. 3.1,

$$\begin{aligned}\tilde{d}_{tc}(C_{ni+1}, ni-1) &= \tilde{d}_{tc}(C_{(ni+1)-1}, (ni-1)-1) + 1 \\ &= \tilde{d}_{tc}(C_{(ni-1)+1}, (ni-1)-1) + 1 \\ &= ni - 1 + 1 + 1 \\ &= ni + 1.\end{aligned}$$

Consequently, $\tilde{d}_{tc}(C_{ni+1}, ni-1) = ni + 1$, for all $n \in N$ and $i \geq 3$.

(viii) To demonstrate, $\sum_{n=i}^{i+2} \tilde{d}_{tc}(C_n, i) = \sum_{n=i-1}^{i+1} \tilde{d}_{tc}(C_n, i-1) + \sum_{n=1}^2 \tilde{d}_{tc}(C_n, n)$, for all $i \geq 3$. The proof proceeds by induction on i . Initially, consider $i = 3$. We obtain

$$\sum_{n=3}^{3+2} \tilde{d}_{tc}(C_n, 3) = \sum_{n=3-1}^{3+1} \tilde{d}_{tc}(C_n, 3-1) + \sum_{n=1}^2 \tilde{d}_{tc}(C_n, n)$$

From Table: $\sum_{n=3}^5 \tilde{d}_{tc}(C_n, 3) = \tilde{d}_{tc}(C_3, 3) + \tilde{d}_{tc}(C_4, 3) + \tilde{d}_{tc}(C_5, 3) = 1 + 4 + 5 = 10$.

From Formula: $\sum_{n=2}^4 \tilde{d}_{tc}(C_n, 2) + \sum_{n=1}^2 \tilde{d}_{tc}(C_n, n) = \tilde{d}_{tc}(C_2, 2) + \tilde{d}_{tc}(C_3, 2) + \tilde{d}_{tc}(C_4, 2) + \tilde{d}_{tc}(C_1, 1) + \tilde{d}_{tc}(C_2, 2) = 1 + 3 + 4 + 1 + 1 = 10$.

It follows that, for $i = 3$, the obtained result is valid. we further establish the validity of the result for $i = k$. Suppose, inductively, that the result is valid for every $j < i, j \in \mathbb{N}$. Using inductive assumption and theorem. 3.1,

$$\begin{aligned} \sum_{n=k}^{k+2} \tilde{d}_{tc}(C_n, k) &= \sum_{n=k}^{k+2} \tilde{d}_{tc}(C_{n-1}, k-1) + 1 \\ &= \sum_{n=k-1}^{k+1} \tilde{d}_{tc}(C_{n-1}, k-2) + 1 \\ &= \sum_{n=k-1}^{k+1} \tilde{d}_{tc}(C_n, k-1) + 1 + 1 \\ &= \sum_{n=k-1}^{k+1} \tilde{d}_{tc}(C_n, k-1) + \sum_{n=1}^2 \tilde{d}_{tc}(C_n, n) \end{aligned}$$

Consequently, $\sum_{n=i}^{i+2} \tilde{d}_{tc}(C_n, i) = \sum_{n=i-1}^{i+1} \tilde{d}_{tc}(C_n, i-1) + \sum_{n=1}^2 \tilde{d}_{tc}(C_n, n)$, for all $i \geq 3$.

(ix) We have, $S_n = \sum_{i=n-2}^n \tilde{d}_{tc}(C_n, i)$. Using theorem. 3.1, we obtain

$$\begin{aligned} S_n &= \sum_{i=n-2}^n \tilde{d}_{tc}(C_{n-1}, i-1) + 1 \\ &= \sum_{i=n-2-1}^{n-1} \tilde{d}_{tc}(C_{n-1}, i) + 1 + 1 \\ &= S_{n-1} + S_2 + S_1 \end{aligned}$$

Consequently, $S_n = \sum_{i=n-2}^n \tilde{d}_{tc}(C_n, i)$ thus, for all $n \geq 5$, $S_n = S_{n-1} + S_1 + S_2$, having initial values $S_1 = 1, S_2 = 1, S_3 = 4, S_4 = 9$.

CONCLUSION

This paper discusses and analyses the total outer- connected dominating sets and total outer- connected domination polynomial of cycle graph. Using the recursive formula and exact formula we constructed the polynomial $\tilde{D}_{tc}(C_n, i) = \sum_{i=\tilde{\gamma}_{tc}(C_n)}^{|\tilde{V}(C_n)|} \tilde{d}_{tc}(C_n, i) x^i$, which we call total outer- connected domination polynomial of C_n and obtain some properties of this polynomial.

ACKNOWLEDGEMENT

The authors would like to thank the anonymous referee for his or her valuable suggestions that have improved the original paper.

REFERENCES

1. Abel Cabera Martinez, Juan Carlos Hernandez Gomez, Ernesto Parra Inza, Jose Maria Sigarreta Almira, On the total outer K-independent domination number of graphs, *Mathematics*, 8(2), (2020).
2. I.Beaton, On dominating sets and domination polynomial, library- archives.Canada.ca, (2021).
3. P. Bosch, E.para Inza, I.Rios Villamar, Total outer- independent domination number:Bounds and algorithms, *Algorithms*, 18(3), (2025).
4. A. Cabrera Martinez, Total outer- independent domination in regular graphs, *Filomat*, 38:18, (2024),6581-6586.
5. B. Chaluvvaraju, N.Majunath, The co-connected domination polynomial of a graph, 18, (2017), 57-70.
6. J. Cyman, The outer connected domination number of a graph, *Australasian journal of combinatorics* 38, (2007), 35-46.

7. J. Cyman, Total outer connected domination number in trees, *Discuss. Math. Graph theory*, 30, (2010), 377-383.
8. J. Cyman, J. Raczek, Total outer-connected domination numbers of trees, *Discrete Appl. Math.*, 157, (2009), 3198-3202.
9. Farshad Kazemnejad, Behanz Pahlavsay, Elisa Palezzato, Michele Torielli, Total outer- connected domination number of middle graphs, *arxiv:2308.02127v1*, (2023).
10. O. Favaron, H. Karami, S.M. Sheikholeslami, On the total outer-connected domination in graphs, *Journal of combinatorial optimization*, 27, (2014), 451-461.
11. Frank Harary, *Graph Theory*, Addison-Wesley Publishing Company, Inc.
12. J. Gross and J. Yellen, *Handbook of Graph Theory*, CRC Press (2004).
13. T. W. Haynes, S. T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998. and *Maths*, 2, (2013), 267-272.
14. Ihsan. A. Alwan, Ahmed.A. Omran, Domination polynomial of the composition of complete graph and star graph, *Journal of physics, Conference series*. 1591, (2020).
15. Johannes. H. Hattingh, Ernst.J. Joubert, T.W. Haynes, A note on the total outer-connected domination number of a tree, *AKCE International journal of Graphs and Combinatorics*, 2, (2010), 223-227.
16. K. Lal Gipson, Subha T, Secure Domination Polynomials Of Paths, *International Journal of Scientific and Technology Research*, 9 (2), (2020), 6517-6521.
17. Nader Jafari Rad, Lutz Volkman, Generalization of the total outer- connected domination in graphs, *RAIRO- Operations Research*, 50, (2016), 233-239.
18. Nasrin Jafari, Saied Alikhani, On the roots of total domination polynomial of graphs, (2019), 795-807.
19. B.S. Panda, Arti Pandey, Complexity of total outer- connected domination problem in graphs, *Discrete applied mathematics*, 199, (2016), 110-122.
20. Saeid Alikhani and Y.H. Peng, Dominating Sets and Domination Polynomials of Paths, *International Journal of Mathematics and Mathematical Sciences*, 542040, (2009).
21. B. Senthil Kumar, H. Naresh Kumar, Y.B. Venkatakrishnan, Total outer- connected vertex edge domination, *Discrete mathematics algorithms and application*, 15(1), (2023).
22. A. Vijayan and K Lal Gipson, Dominating Sets of Square of Centipedes, *International Journal of Mathematical Archive*, 4(2), (2013), 298-310.
23. A. Vijayan and K. Lal Gipson, Dominating Sets and Domination Polynomials of Square of Paths, *Open Journal of Discrete Mathematics*, 3, (2013), 60-69.