Research Article

Fuzzy Graph-Theoretic Reliability and Fault Tolerance in Electrical Networks

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Abstract:

Reliability and fault tolerance represent important factors that should be considered in the process of designing and studying electrical networks. Graph-theoretic approaches allow one to describe network elements as vertices and their connections as edges. In case if there is some level of uncertainty in relation to the conditions of components and connections between them, there may be a need in a more flexible approach to modelling. Fuzzy graph theory represents an approach where this uncertainty can be expressed using membership values of vertices and edges. This paper aims at developing a model based on fuzzy graph theory that would be used to investigate structural reliability and fault tolerance of electrical networks. It will take into account fuzzy connectivity, path strength, critical elements and structural transformations due to the deletion of vertices and edges. Mathematical statements and theorems would be suggested to analyze the relations between these concepts and maintaining of network connectivity after element failures. Several examples of hypothetical electrical networks would be used to illustrate the proposed concepts and compare different network structures. The research is focused on the analysis of structural reliability in the context of fuzzy models and it does not consider failure probability and real electrical characteristics.

Keywords: Fuzzy graph theory, electrical network reliability, fault tolerance, fuzzy connectivity, path strength, network vulnerability, critical components, structural resilience.

1 Introduction:

Electrical networks are formed of interconnected components, including substations, transmission lines, distribution feeders, and switches. Failure of any component can lead to a loss of connectivity or a reduction of the network's ability to serve connected loads. As a result, reliability and fault tolerance are important areas of investigation in the analysis of electrical networks. Identifying vulnerable components and investigating the response of networks to failures is key to evaluating the structural reliability of a network.

Graph theory offers one approach to the analysis of electrical networks. A graph representation encodes vertices as locations or components within the network, and edges represent connections between them. Connectivity, paths, and cut sets are amongst the properties used in graph theory to evaluate whether two locations in a graph will remain connected in the event of the failure of a specific component. However, a conventional graph generally captures a notion of connectivity which considers connections as a binary presence or absence. Whilst such a representation is useful, it does not account for uncertainties in the condition of components or the suitability of connections for a specific assessment.

Fuzzy set theory, introduced by Zadeh [1], offers an alternative approach to capturing uncertainty. Rosenfeld [2] introduced the notion of fuzzy graphs, which extend the graph-theoretic modelling approach by encoding membership values for vertices and edges[7]. Further advances in the fuzzy graph theory have provided alternative means of investigating connectivity and related properties [3], [6]. These methods offer a potential alternative approach to the analysis of electrical networks, where information on the condition of components or the suitability of connections is uncertain or imprecise.

In the field of reliability engineering, fuzzy methods have been investigated as a means of handling uncertainties in the context of safety and reliability assessment [4]. Graph-based reliability analysis also considers the impact of network structure and alternative means of connections on the impacts of component failure [5]. Combining these areas of investigation presents an interesting challenge: how can fuzzy graph properties be used to evaluate the structural ability of an electrical network to retain connectivity after the removal of a vertex or an edge? This challenge is associated with the need to provide clear definitions of fuzzy connectivity, path strength, and the impact of component removal.

In this paper, a fuzzy graph-theoretic framework is proposed to investigate structural reliability and fault tolerance in electrical networks. The approach encodes the network using vertex and edge membership values,

and examines the impact of component removal on connectivity. Particular emphasis is placed on alternative paths, critical components, and the changes in the strength of connections between network locations. The framework is proposed for use in a structural assessment scenario, under explicitly defined fuzzy modelling assumptions. Fuzzy membership values are therefore viewed as degrees of membership, rather than as approximations of component failure probabilities.

The goals of this study are to formulate a fuzzy graph of an electrical network, define suitable measures for path strength and connectivity, and analyze their variations when a vertex or an edge fails. It is intended to develop mathematical relations of fuzzy graph properties with structural vulnerability and fault tolerance. Hypothetical examples of electrical networks are given to demonstrate the definitions and calculations. Examples provide a mathematical comparison of network configurations, without needing experimental verification. The rest of the paper is organized as follows. Section 2 presents preliminary definitions and graph concepts. Section 3 introduces the fuzzy graph model for electrical network reliability, while Section 4 analyzes fault tolerance and network vulnerability. Section 5 develops the main mathematical results and their proofs, and Section 6 provides illustrative examples and comparison. Finally, Section 7 concludes the results and presents possible directions of further research.

2 Preliminaries and Basic Definitions

This section presents the graph-theoretic concepts and fuzzy graph definitions that are necessary for the work. The definitions describe the mathematical representation of electrical networks and the analysis of the effects of component failures on network connectivity.

2.1 Basic Graph Concepts

A graph is an ordered pair $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ where V is a set of vertices, which are a finite, nonempty set. The edges represent the connections between the vertices, and they form a set E. In an electrical network, the vertices can be buses, the edges can be transmission lines, and feeders.

A path between two vertices is a sequence of distinct vertices in which consecutive vertices are connected by edges. A graph is connected if a path exists between every pair of distinct vertices. An edge whose removal disconnects a connected graph is called a bridge. A vertex whose removal increases the number of connected components is called an articulation vertex.

These concepts help identify components that are important for maintaining network connectivity. However, a classical graph does not directly represent uncertainty in the condition or suitability of its components. Fuzzy graph theory provides an extension by assigning membership values to vertices and edges [2], [3].

2.2 Fuzzy Sets

A fuzzy set assigns each element of a universe a membership value between zero and one. This concept was introduced by Zadeh [1] to represent situations in which membership is not limited to a binary classification.

Let X a nonempty set. A fuzzy set A on X is characterized by the membership function

$$\mu_{\mathrm{A}}: \mathrm{X} \to [0,1].$$

For each $\mathrm{x} \in \mathrm{X}$, the value $\mu_{\mathrm{A}}(\mathrm{x})$ represents the degree of membership of x in A. A value of zero indicates no membership, whereas a value of one indicates full membership.

In an electrical network, membership values may represent an assessment of component condition or connection adequacy, provided the assessment criterion is clearly defined. These values should not automatically be interpreted as probabilities of successful operation.

2.3 Fuzzy Graphs

Rosenfeld [2] extended graph theory by introducing fuzzy membership values for vertices and edges.

Definition 2.1 (Fuzzy graph). Let V be a finite, nonempty set. A fuzzy graph is a triple $\mathrm{G} = (\mathrm{V}, \sigma, \mu)$, where $\sigma: \mathrm{V} \to [0,1]$ is the vertex membership function and $\mu: \mathrm{V} \times \mathrm{V} \to [0,1]$ is the edge membership function satisfying

$$\mu(x, y) \leq min\{\sigma(x), \sigma(y)\}$$

for all $x, y \in V$. For an undirected fuzzy graph, $\mu(x, y) = \mu(y, x)$, and $\mu(x, x) = 0$.

This condition ensures that the membership value of an edge does not exceed the membership value of either endpoint. In the present study, these functions provide the mathematical representation of network components and their connections.

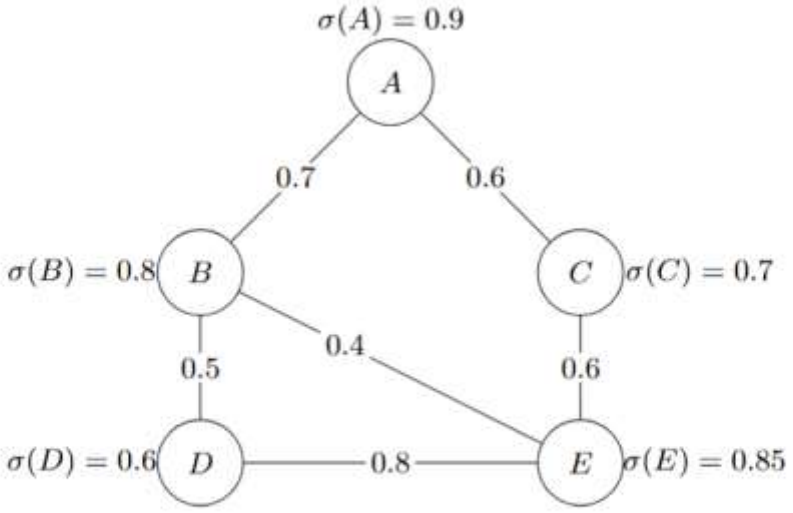


Figure 1: A fuzzy graph illustrating vertex and edge membership values.

2.4 Paths and Path Strength

Definition 2.2 (Path strength). Let

$$P = (v_0, v_1, \dots, v_k)$$

be a path in a fuzzy graph G. Its strength is defined as the minimum membership value among its edges:

$$S(P) = \min_{1 \le i \le k} \mu(v_{i-1}, v_i).$$

This definition assigns the path a strength determined by its weakest edge. It is useful when comparing alternative paths between network locations under a minimum-based fuzzy graph model [2], [3].

2.5 Fuzzy Connectivity

Definition 2.3 (Fuzzy connectivity). Let $x, y \in V$ be distinct vertices. The fuzzy connectivity between x and y is defined as

$$C_G(x, y) = \max_{P \in \mathcal{P}_{xy}} S(P),$$

Where $\mathcal{P}_{xy}$ denotes the set of paths connecting x and y

If no path connects the two vertices, define $C_G(x, y) = 0$. For a finite graph with at least one path between the vertices, the maximum exists.

This measure identifies the strongest path between two vertices according to the adopted definition of path strength. It provides a structural comparison of network connectivity, but it is not, by itself, a probability of successful power delivery.

2.6 Cut Sets and Critical Components

Definition 2.4 (Edge cut set). Let $G = (V, E)$ be a connected graph. A set $F \subseteq E$ is an edge cut set if removing all edges in F disconnects the graph. The cut set is minimal if no proper subset of F disconnects the graph.

Cut sets help identify combinations of connections whose failure can divide an electrical network into disconnected parts.

Definition 2.5 (Critical edge). Let $x, y \in V$ be distinct vertices connected in G. An edge $e \in E$ is critical with respect to the pair (x, y) if

$$C_{G-e}(x, y) < C_G(x, y),$$

Where $G - e$ denotes the graph obtained by removing edge e.

This definition identifies an edge whose removal reduces the fuzzy connectivity between a specified pair of vertices. An edge may be critical for one pair of vertices but not for another, depending on the available alternative paths.

2.7 Electrical Network Representation

An electrical network can be represented by a graph, in which vertices stand for buses, substations, or distribution points, and edges stand for transmission lines, feeders, or switching connections. The network topology defines the structural paths between these locations, and determines the consequences of component failures [5].

In this study, the network is initially represented as a finite, undirected fuzzy graph. Vertex and edge membership values are attributed in accordance with explicitly defined modelling criteria. The analysis focuses on structural connectivity and its changes following component removal.

Electrical operating constraints, such as voltage limits, thermal limits, and power-flow feasibility, are not captured by graph connectivity alone. Therefore, the proposed framework should be interpreted as a structural assessment rather than a complete electrical or probabilistic reliability analysis.

3 Fuzzy Graph Model of Electrical Network Reliability

3.1 Representation of an Electrical Network

An electrical network consists of interconnected components such as buses, transmission lines, feeders, transformers, and switching devices. The ability of the network to maintain a connection between two locations depends not only on the presence of individual components but also on the availability of alternative paths between them.

To study this structural property, consider an electrical network represented by a fuzzy graph

$$G = (V, \sigma, \mu),$$

Where V is the set of vertices, σ assigns a membership value to each vertex, and μ assigns a membership value to each edge. The vertices represent network locations, such as buses or substations, whereas the edges represent connections between two locations.

The membership values are contained in the range $[0,1]$. The values, which are close to 1, indicate a higher grade of membership in the corresponding structural relationship, in comparison with the values that are close to 0. The values should be interpreted on the basis of the specified modeling criterion (not necessarily the probability of a component's survival or failure and electrical performance measures).

For a vertex v∈V, its membership value is denoted by σ(v).For an edge $e = uv \in E$, its membership value is denoted by $\mu(u, v)$. The standard fuzzy graph constraint is

$$\mu(u, v) \leq min\{\sigma(u), \sigma(v)\}.$$

This condition guarantees that the degree of membership of any edge in a fuzzy graph is not greater than the degree of membership of each of the two vertices which this edge connects.

3.2 Assigning Membership Values to Network Components

These values should be assigned using some well-defined criterion such as a measure of structural importance or a confidence level in the existence of the connection or a normalized measure of some other information relevant to the particular problem. Once a criterion is chosen for a particular application, it must be consistently used in defining the membership values.

Let v be a bus and e be a transmission line. The values $\sigma(v)$ and $\mu(e)$ describe their respective membership levels in the proposed fuzzy model. They do not, by themselves, provide a quantitative prediction of how frequently the bus or line will fail.

This distinction is important because conventional reliability analysis often uses probability-based quantities, whereas fuzzy graph theory represents degrees of membership. A probabilistic reliability model and a fuzzy graph model may be used together, but a conversion between their values requires an explicit mathematical justification.

For the present framework, the membership values are used to examine the strength of network connections and the effect of component removal on structural connectivity.

3.3 Path Strength in an Electrical Network

Consider two vertices $s, t \in V$ representing the source and destination buses. A path connecting them can be written as

$$P = (v_0, v_1, \dots, v_k),$$

Where $v_0 = s$, $v_k = t$, and each consecutive pair of vertices is connected by an edge.

The strength of the path is determined by its weakest edge membership value:

$$S(P) = \min_{1 \leq i \leq k} \mu(v_{i-1}, v_i).$$

This definition embodies a bottleneck principle. A path with several edges with high membership values may have a relatively low strength if one of its edges has a low membership value. For example, if a path has edges with membership values 0.8,0.6, and 0.9, its strength is

$$S(P) = min\{0.8,0.6,0.9\} = 0.6.$$

The value 0.6 represents how strong this path is within the selected fuzzy model. It should not be misconstrued to mean there's a 60% chance that this path will operate successfully.

3.4 Fuzzy Connectivity Between Network Locations

A pair of buses may be connected by more than one path. Therefore, examining a single path is insufficient to describe the overall structural connection between them.

Let $\mathcal{P}_{st}$ denote the collection of all paths between s and t. The fuzzy connectivity between these vertices is defined as

$$C_G(s,t) = \max_{P \in \mathcal{P}_{st}} S(P).$$

If no path connects s and t, then $C_G(s,t) = 0$.

This measure identifies the strongest available path between two network locations according to the assigned edge membership values. Unlike a measure based on the sum of path strengths, the maximum–minimum formulation evaluates each path by its weakest edge and selects the path with the highest resulting value.

Suppose two alternative paths between a source bus and a destination bus have strengths 0.5 and 0.7, respectively. Their fuzzy connectivity is

$$C_G(s,t) = max\{0.5,0.7\} = 0.7.$$

The existence of the second path provides a stronger structural connection under the selected membership model.

3.5 A Structural Measure of Network Reliability

In this paper, network reliability is studied through the preservation of fuzzy connectivity when components are removed. This provides a structural measure of how strongly two network locations remain connected under a specified change in network structure.

Let $e \in E$ be an edge. The graph obtained by removing e is denoted by $G - e$. The change in fuzzy connectivity between s and t following the removal of this edge is defined as

$$\Delta_e(s,t) = C_G(s,t) - C_{G-e}(s,t).$$

Because removing an edge cannot increase the maximum–minimum connectivity between a fixed pair of vertices, this difference is non-negative:

$$\Delta_e(s,t) \geq 0.$$

A positive value indicates that removing the edge reduces the fuzzy connectivity between the selected vertices. A zero value indicates that the removal does not reduce their fuzzy connectivity, although it may still affect other pairs of vertices or other network properties.

For example, if the original fuzzy connectivity is 0.8 and the connectivity after edge removal is 0.5, then

$$\Delta_e(s,t) = 0.8 - 0.5 = 0.3.$$

The value 0.3 represents the decrease in fuzzy connectivity, not the loss of 30% in probabilistic reliability.

The same procedure can be used to a vertex v, if v is not one of the two vertices to which connectivity is being calculated. The graph resulting from the removal of v from G is denoted by $G - v$. Define

$$\Delta_v(s,t) = C_G(s,t) - C_{G-v}(s,t).$$

These measures form a basis for identifying components whose removal has a significant effect on the structural connectivity of an electrical network.

3.6 Scope of the Proposed Model

The introduced model is based on the topological structure of the studied electrical network and the membership values of its elements. Thus, the proposed approach is mathematically convenient for analyzing alternative paths and studying the impact of removing edges or vertices.

However, the model is not focused on the electrical processes in the network; hence, it cannot consider power-flow equations, voltage restrictions, thermal losses, protection coordination schemes, and dynamic stability. In other words, when applying the model to real networks, one should be aware that the results concern only topological aspects of the networks. To consider electrical constraints, additional assumptions and data would be necessary for this model.

Therefore, the definitions listed above are the starting point for further research concerning the network's ability to withstand various failures, including cascading ones.

4 Fault Tolerance and Network Vulnerability

4.1 Concept of Fault Tolerance

Fault tolerance is a network's ability to maintain its essential connectivity after the failure of one or more components. In the context of an electric network, a fault in a transmission line, feeder, or switching connection can interrupt the connection between two buses. The impact of the fault depends on the network's topology and the availability of alternate paths.

A network with multiple alternate paths can maintain its connectivity after the removal of a particular edge while the one depending on a single connection can have drastically reduced connectivity after the removal of the connection.

Within the fuzzy graph framework, the ability of a network to withstand faults is explored by comparing the fuzzy connectivity of the original graph and the one after the removal of components. Let $G = (V, \sigma, \mu)$ be a fuzzy graph and let $s, t \in V$ be two distinct vertices. The reduction in connectivity caused by removing an edge e is

$$\Delta_e(s,t) = C_G(s,t) - C_{G-e}(s,t).$$

A smaller reduction indicates that the initial level of fuzzy connectivity is better preserved for the chosen pair of vertices. Nevertheless, this value reflects the property of the structural connectivity under the given membership values, not the probability of an uninterrupted electrical service.

4.2 Critical Edges and Critical Vertices

A component is particularly important when its removal reduces the connectivity between selected network locations. This importance can be examined using the following definitions.

Definition 4.1 (Critical edge). An edge $e \in E$ is critical with respect to vertices s and t if

$$C_{G-e}(s,t) < C_G(s,t).$$

Thus, an edge is critical for a particular vertex pair when its removal strictly decreases the fuzzy connectivity between that pair.

Definition 4.2 (Critical vertex). A vertex $v \in V \setminus \{s,t\}$ is critical with respect to s and t if

$$C_{G-v}(s,t) < C_G(s,t).$$

Here, $G - v$ denotes the graph obtained by removing v and its incident edges.

These definitions are pair-dependent. An edge may be critical for one source–destination pair but not for another. Similarly, a vertex may have a substantial effect on connectivity in one part of a network while having little effect elsewhere.

The definitions also distinguish fuzzy connectivity from ordinary graph connectivity. A component may be removed without disconnecting the graph in the usual binary sense, yet its removal may still reduce the maximum–minimum connectivity value.

4.3 Fuzzy Connectivity Loss

To compare the effect of different component removals, define the connectivity loss associated with an edge e by

$$L_e(s,t) = C_G(s,t) - C_{G-e}(s,t).$$

For a vertex $v \notin \{s,t\}$, define

$$L_v(s,t) = C_G(s,t) - C_{G-v}(s,t).$$

Both quantities are non-negative under the maximum–minimum connectivity definition. A larger value indicates a greater loss of fuzzy connectivity for the specified vertex pair.

If $C_G(s,t) = 0.9$ and the removal of an edge gives $C_{G-e}(s,t) = 0.4$, then

$$L_e(s,t) = 0.9 - 0.4 = 0.5.$$

If removing a different edge leaves the connectivity unchanged at 0.9, its connectivity loss is zero. The first edge therefore has a greater structural effect on the connection between s and t under this model.

For comparisons involving different vertex pairs, the pair being studied should be stated explicitly. A network-wide assessment may require examining several source–destination pairs rather than relying on a single connectivity-loss value.

4.4 Alternative Paths and Fault Tolerance

The presence of alternative paths is an important factor in determining whether a network can preserve connectivity after a component is removed. Consider two vertices s and t connected by two paths, P_1 and P_2, with strengths $S(P_1)$ and $S(P_2)$.

Their fuzzy connectivity is

$$C_G(s,t) = max\{S(P_1), S(P_2)\},$$

provided these are the strongest paths in the graph.

If an edge belonging only to P_1 is removed, P_2 may continue to provide a connection between s and t. If the edge belongs to every strongest path, its removal may reduce the fuzzy connectivity value. The extent of the reduction depends on the strengths of the remaining paths.

This observation is critical. The number of alternative paths is not the sole factor determining fuzzy fault tolerance. The membership values of the edges on these alternative paths also affect the connectivity that is maintained after a fault occurs.

4.5 A Proposition on Connectivity Loss

Proposition 4.1. Let G be a fuzzy graph, and let s, t be the two distinct vertices. Then for any edge e whose removal leaves s and t in the vertex set,

$$C_{G-e}(s,t) \leq C_G(s,t).$$

Consequently,

$$L_e(s,t) \geq 0.$$

Proof. Every path connecting s and t in $G-e$ is also a path in G, because $G-e$ is obtained by deleting an edge without adding any new edges. Therefore, the set of paths available in $G-e$ is a subset of the paths available in G.

The strength of each remaining path is unchanged because the membership values of the edges not removed remain the same. Since fuzzy connectivity is defined as the maximum strength among all available paths, taking the maximum over the smaller set cannot produce a value greater than the maximum over the original set. Hence,

$$C_{G-e}(s,t) \leq C_G(s,t).$$

Rearranging gives

$$C_G(s,t) - C_{G-e}(s,t) \geq 0,$$

which proves that $L_e(s,t) \geq 0$. □

4.6 Network Vulnerability

Network vulnerability describes the extent to which the structural performance of a network is affected by the loss of components. Within the present framework, vulnerability can be assessed through the connectivity losses associated with edge or vertex removal.

For a fixed pair s,t, a component producing a large connectivity loss can be considered structurally important for that connection. Components that produce no loss under the selected measure are less influential for that particular pair, although they may remain important for other pairs or for electrical operating conditions not represented in the model.

For a finite set of candidate edges $E' \subseteq E$, a simple pair-specific vulnerability indicator is given by

$$V_{s,t}(G;E') = \max_{e\in E'} L_e(s,t).$$

This quantity identifies the largest connectivity loss among the candidate edge removals being considered. If E' is empty, the indicator is left undefined.

The indicator is useful for comparing the sensitivity of a network to individual edge removals. It should not be interpreted as a universal measure of electrical-system vulnerability, because it depends on the selected vertex pair, candidate components, and membership assignments.

The definitions and results in this section provide a foundation for the mathematical analysis of connectivity preservation, critical components, and structural fault tolerance.

5 Main Results and Mathematical Analysis

This section presents the mathematical results related to fuzzy connectivity, alternative path, and removal of components. The results are obtained from the developed maximum–minimum definition of fuzzy connectivity proposed in Sect. 2 and the structural model established in Sects. 3 and 4.

5.1 Connectivity Under Edge Removal

The first result establishes that removing an edge cannot increase the fuzzy connectivity between two vertices.

Proof. Any path from s to t is contained in $G-e$ is also a path in G. The removal of an edge can eliminate paths, but it cannot introduce new paths. Since the strength of each remaining path is unchanged, the maximum strength among the paths in $G-e$ cannot exceed the maximum strength among the paths in G.

Therefore,

Corollary 5.1. The connectivity loss caused by removing an edge is non-negative:

$$L_e(s,t) = C_G(s,t) - C_{G-e}(s,t) \geq 0.$$

Proof. The result follows directly from Theorem 5.1 by subtracting $C_{G-e}(s,t)$ from both sides. □

This result provides a basic consistency condition for the proposed model. If a calculation gives a negative connectivity loss after deleting an edge, the path strengths or connectivity values have been evaluated incorrectly.

5.2 Connectivity Through Alternative Paths

Alternative paths can preserve fuzzy connectivity when an edge is removed. The following theorem describes the case in which a path remains available with the same strength as the original fuzzy connectivity.

Theorem 5.2. Let G be a fuzzy graph and let $s,t \in V$ be distinct vertices. Suppose there exists an s-t path P that does not contain an edge e, and that

$$S(P) = C_G(s,t).$$

Then

$$C_{G-e}(s,t) = C_G(s,t).$$

Proof. Since P does not contain e, the path remains in $G - e$. Therefore,

$$C_{G-e}(s,t) \geq S(P) = C_G(s,t).$$

On the other hand, Theorem 5.1 gives

$$C_{G-e}(s,t) \leq C_G(s,t).$$

Combining these inequalities yields

$$C_{G-e}(s,t) = C_G(s,t).$$

Hence the removal of e does not reduce the fuzzy connectivity between s and t. □

Interpretation. If at least one path attaining the original maximum connectivity remains after an edge is removed, the connectivity value is preserved. This result provides a sufficient condition for zero connectivity loss. It does not require every alternative path to have the same strength.

5.3 Characterization of a Critical Edge

The next result gives a more precise characterization of when an edge is critical under the maximum–minimum connectivity measure.

Theorem 5.3. Let G be a finite fuzzy graph, and let $s, t \in V$ be distinct vertices with $C_G(s,t) > 0$.

An edge $e \in E$ is critical with respect to s and t if and only if every s-t path whose strength equal $C_G(s,t)$ contains e.

Proof.

Suppose first that every s-t path attaining the maximum strength contains e. After removing e, none of these maximum-strength paths remains. Since G is finite, there are only finitely many simple s-t paths. Consequently, if any paths remain, the maximum strength among the remaining paths must be strictly less than $C_G(s,t)$. If no path remains, the new connectivity is zero, which is also strictly less than $C_G(s,t)$ by the hypothesis that $C_G(s,t) > 0$. Thus,

$$C_{G-e}(s,t) < C_G(s,t),$$

And e is critical.

Conversely, suppose that e is critical. Then

$$C_{G-e}(s,t) < C_G(s,t).$$

If there were a maximum-strength s-t path not containing e, that path would remain in $G - e$ and would retain strength $C_G(s,t)$. This would imply

$$C_{G-e}(s,t) \geq C_G(s,t),$$

contradicting the strict inequality above. Thus every maximum-strength s-t path contains e. □

Remark 5.1. The finiteness assumption ensures that the maximum path strength is attained. The condition $C_G(s,t) > 0$ excludes the trivial case in which the original connectivity is already zero.

This characterization shows that an edge is critical precisely when it is contained in each path achieving the maximum fuzzy connectivity between the given vertices.

5.4 A Bound on Connectivity Loss

The amount by which the fuzzy connectivity decreases upon removal of an edge is removed is also bounded by the original connectivity.

Theorem 5.4. For any edge $e \in E$ and any distinct vertices $s, t \in V$,

$$0 \leq L_e(s,t) \leq C_G(s,t).$$

Proof. By Theorem 5.1,

$$C_{G-e}(s,t) \leq C_G(s,t).$$

Therefore,

$$L_e(s,t) = C_G(s,t) - C_{G-e}(s,t) \geq 0.$$

Moreover, fuzzy connectivity is non-negative, so $C_{G-e}(s,t) \geq 0$. It follows that

$$C_G(s,t) - C_{G-e}(s,t) \leq C_G(s,t).$$

Thus,

$$0 \leq L_e(s,t) \leq C_G(s,t).$$

And so we have proved the theorem. □

The upper bound shows that the connectivity loss cannot be more than the original connectivity value. In particular, if removing an edge disconnects s from t, then $C_{G-e}(s,t) = 0$, and the connectivity loss equals $C_G(s,t)$.

5.5 Extension to a Set of Removed Edges

In practice, networks can experience multiple edge removals simultaneously, and hence, the framework of the present section can be generalized to the case of an arbitrary edge set $F \subseteq E$. By analogy with definition 5.2, let us denote by $G - F$ the graph obtained from G by removing all edges in F.

Proposition 5.1. For any edge set $F \subseteq E$ and any distinct vertices $s, t \in V$,

$$C_{G-F}(s,t) \leq C_G(s,t).$$

Proof.: Every path remaining in $G - F$ is also a path in G, and the strength of each remaining path is unchanged. Hence the maximum path strength in $G - F$ cannot exceed the maximum path strength in G. Therefore,

$$C_{G-F}(s,t) \leq C_G(s,t).$$

Thus the result follows. □

Define the connectivity loss associated with the edge set F by

$$L_F(s,t) = C_G(s,t) - C_{G-F}(s,t).$$

The proposition ensures that $L_F(s,t) \geq 0$. This formulation can be used to examine multiple-edge outages or planned component removals, provided that the analysis remains within the structural assumptions of the fuzzy graph model.

5.6 Implications for Electrical Network Analysis

The results established in this section demonstrate that the framework we propose possesses some desirable properties:

1. The removal of edges cannot increase fuzzy connectivity.
2. A surviving path which achieves the original maximum strength is sufficient to ensure the preservation of the connectivity value.
3. An edge is critical for a vertex pair if and only if it belongs to all the maximum-strength paths connecting this vertex pair, under the above-mentioned finite-graph and positive-connectivity assumptions.
4. The loss of connectivity cannot be negative or greater than the original connectivity.
5. The same monotonicity property holds when multiple edges are removed at once.

These properties can be used to compare the criticality of network elements. They do not, however, characterize the network's reliability in terms of a probability of failure, nor do they ensure the maintenance of electrical power service under certain operating conditions. These are additional questions which require further data and analysis beyond the mere possession of a fuzzy connectivity measure.

6 Illustrative Examples and Comparative Analysis

This section will demonstrate the proposed fuzzy graph model through a hypothetical electrical network. This case study explains how path strength, fuzzy connectivity, and connectivity loss can be calculated and used to find structurally critical connections. The values are examples of membership values and are not intended to reflect any electrical reliability measurements.

6.1 Description of the Network

Consider a network represented by the fuzzy graph $G = (V, \sigma, \mu)$, where

$$V = \{A, B, C, D, E\}.$$

Let the vertex membership values be

$$\sigma(A) = 0.9, \quad \sigma(B) = 0.8,$$
$$\sigma(C) = 0.7, \quad \sigma(D) = 0.6,$$
$$\sigma(E) = 0.85.$$

Suppose the network contains the following edges and their membership values:

Edge	Membership value
AB	0.7
AC	0.6
BD	0.5
CE	0.6
DE	0.8
BE	0.4

Each edge membership value satisfies the fuzzy graph constraint $\mu(u,v) \leq min\{\sigma(u), \sigma(v)\}$.
For example,

$$\mu(A,B) = 0.7 \leq min\{0.9,0.8\} = 0.8.$$

The network contains multiple routes between some vertices. These routes allow the effect of removing individual edges to be examined.

6.2 Calculation of Path Strength

Consider the connection between vertices A and E. Three possible paths are:

$$P_1 = (A,B,E),$$
$$P_2 = (A,B,D,E),$$
$$P_3 = (A,C,E).$$

The strength of each path is calculated using the minimum edge membership value along that path.
For P_1,

$$\begin{aligned} S(P_1) &= min\{\mu(A,B), \mu(B,E)\} \\ &= min\{0.7,0.4\} \\ &= 0.4. \end{aligned}$$

For P_2,

$$\begin{aligned} S(P_2) &= min\{\mu(A,B), \mu(B,D), \mu(D,E)\} \\ &= min\{0.7,0.5,0.8\} \\ &= 0.5. \end{aligned}$$

For P_3,

$$\begin{aligned} S(P_3) &= min\{\mu(A,C), \mu(C,E)\} \\ &= min\{0.6,0.6\} \\ &= 0.6. \end{aligned}$$

The strongest of these paths has strength 0.6. Since the listed edges provide all the possible simple paths from A to E in this network, the fuzzy connectivity is

$$C_G(A,E) = max\{0.4,0.5,0.6\} = 0.6.$$

6.3 Effect of Edge Removal

The effect of removing an edge can be evaluated by recalculating the fuzzy connectivity between A and E.
Case 1: Removal of edge AC.
The path $P_3 = (A,C,E)$ is no longer available. The remaining paths have strengths 0.4 and 0.5. Hence,

$$C_{G-AC}(A,E) = max\{0.4,0.5\} = 0.5.$$

The connectivity loss is

$$L_{AC}(A,E) = 0.6 - 0.5 = 0.1.$$

Therefore, removing AC reduces the fuzzy connectivity between A and E by 0.1.
Case 2: Removal of edge BE.
The path $P_1 = (A,B,E)$ is no longer available. The other two paths remain unchanged, giving

$$C_{G-BE}(A,E) = max\{0.5,0.6\} = 0.6.$$

Consequently,

$$L_{BE}(A,E) = 0.6 - 0.6 = 0.$$

The removal of BE does not reduce the fuzzy connectivity between A and E, because the path P_3 still attains the original connectivity value.
Case 3: Removal of edge DE.
The path $P_2 = (A,B,D,E)$ is no longer available. The remaining paths have strengths 0.4 and 0.6. Thus,

$$C_{G-DE}(A,E) = max\{0.4,0.6\} = 0.6,$$

And

$$L_{DE}(A,E) = 0.6 - 0.6 = 0.$$

In this case, the removal of DE also leaves the fuzzy connectivity unchanged.

6.4 Comparative Analysis

The results can be summarized as follows.

Removed edge	Connectivity after removal	Connectivity loss
AC	0.5	0.1
BE	0.6	0.0
DE	0.6	0.0

The comparison shows that the effect of an edge removal depends on the alternative paths available between the selected vertices. In this example, AC is critical with respect to A and E, since its removal reduces the fuzzy connectivity. The edges BE and DE are not critical for this pair under the adopted definition.

This does not mean that BE and DE are unimportant throughout the network. Their removal could affect the connectivity of other vertex pairs. The assessment is therefore specific to the source–destination pair being considered.

6.5 Interpretation for Electrical Networks

In an electrical network, the vertices of this example could represent buses or substations, and the edges could represent transmission lines or feeders. The connectivity-loss measure enables an assessment of the impact of the removal of individual connections on the strongest available structural path between two locations.

The example also highlights the importance of the existence of multiple paths. A connection can be removed without any loss of fuzzy connectivity, provided that another path of equal or greater strength is available. On the other hand, the removal of an edge that is part of all strongest paths will reduce the connectivity value.

The results should be considered with the limitations of the model in mind. They do not demonstrate that power can be transferred through every available path under actual operating conditions, since line capacity, voltage levels, protection schemes and power-flow constraints would have to be evaluated in detail in an engineering study.

7 Conclusion and Future Research

This paper develops a fuzzy graph theory-based technique for measuring structural reliability and fault tolerance of electrical networks by treating network locations as vertices and connections as edges. Membership values were used to represent the degree of connection strengths and structural linkage between network components. Maximum–minimum approach has been used to calculate path strength and fuzzy connectivity between certain vertices.

Mathematical findings proved that deleting an edge can never result in higher fuzzy connectivity and that decrease of connectivity is non-negative. In addition, the analysis has provided the characterisation of critical edges of finite fuzzy graphs having positive connectivity. This gives an insight into the preservation of the original connectivity value due to alternative paths and the impact of some edge deletions on structural linkage between network locations.

An illustration of how the suggested measurements can be applied in a particular example has been provided. The comparison of connectivity before and after the edge deletion allows for the identification of connections having greater impact on the structural linkage between a certain source-destination pair of network locations. This paper has limitations since the fuzzy membership values do not necessarily refer to failure probability and the model itself does not include any consideration of power-flow constraints, line capacities, voltage stability and protection systems. Therefore, the results obtained should be regarded as structural assessment rather than predictions of electrical network reliability.

There are possible directions for future research. Firstly, the values of membership could be derived from the documentation of engineering assessments and/or available data on network operation. Secondly, the model can be combined with probabilistic reliability approach in order to draw a distinction between the degrees of membership and failure probabilities of components. Thirdly, larger networks can be studied in order to compare the effects of removal of single versus several components. Finally, the use of power flow restrictions could help in estimating the relationships between fuzzy connectivity measures and electrical service continuity. In general, fuzzy graph theory is an appropriate mathematical instrument for investigating the preservation of connections and determining critical points. However, the comparison with the traditional reliability approaches and the empirical validation based on the real-life electric network data should be implemented to prove its effectiveness.

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