Research Article

Co-even Dominating sets and Co-even Domination Polynomial of Fan graph F_n

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Abstract

Let G=(V,E) be a simple graph. A set D⊆V is a dominating set of G, if for every vertex in V\D is adjacent to exactly one vertex in D. A subset D of V is called a co-even dominating set if deg(v) is even for all v∈V-D. The set D is called a minimal co-even dominating set if has no proper subset is a co-even dominating set of G. The minimal co-even dominating sets of a graph G is denoted by MCEDS(G). The co-even domination number is denoted by γ_coe (G). Let G be a graph with n vertices and D_coe (G,i) denote the family of all co-even dominating sets of G with cardinality i. In this paper, we construct the co-even domination polynomial of Fan graphs F_n.

Keywords: Co-even dominating set, co-even domination polynomial, Fan Graph F_n.

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by m and n respectively. For basic definitions and terminologies we refer to Buckley et. Harary, 1990. Two vertices u and v are said to be adjacent if uv is an edge of G. The open neighborhood of a vertex v in a graph G is defined as $N_G[v] = N_G(v) \cup \{v\}$. A subset $S \subseteq V(G)$ is called a dominating set if every vertex $v \in V(G)\backslash D$ is adjacent to a vertex $u \in S$. The domination number $\gamma(G)$ of a graph G denotes the minimum cardinality of such dominating sets of G. A minimum dominating set of a graph G is hence often called as γ-set of G in Alikhani et Peng.2009. A dominating set D is called a co-even dominating set if $\deg(v)$ is even number for all $v \in V - D$ and $\gamma_{coe}(G)$ is the minimum cardinality of a co-even dominating set of G. The concept of co-even domination number of a graph is defined by M.M. Shalaan and A.A. Omran. In this we introduce the concepts of co-even domination polynomials of Fan Graph F_n.

The Fan graph F_n can be constructed by joining the path and complete graph of order one and denoted by $F_n \equiv P_n + K_1$. Let $\gamma_{coe}(G)$ be the domination number of G. Let $D_{coe}(G, i)$ denote the family of all co-even dominating sets of G with cardinality i. Let $d_{coe}(G, i) = |D_{coe}(G, i)|$. Then the co-even domination polynomial of G is

$$\boldsymbol{D_{coe}(G, x) = \sum_{i=\gamma_{coe}(G)}^{n} d_{coe}\,(G, i)\, x^i}$$

Example 1.1 Consider the graph F_8.

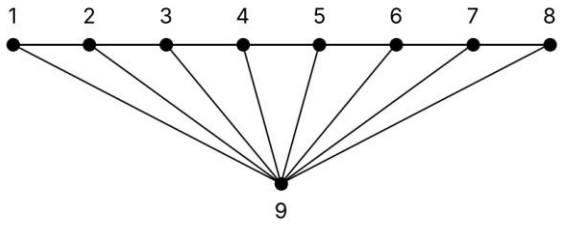

Figure 1.1

In Fig 1.1, $D = \{3,4,5,6,7,2\}$ is a co-even dominating set. For $V - D = \{1,8,9\}$ the vertex $1 \in V - D$ has even degree, the vertex $8 \in V - D$ has even degree and the vertex $9 \in V - D$ has even degree. So the set $\{3,4,5,6,7,2\}$ is a co-even dominating set.

In the next section, we obtain some properties of co-even dominating sets of F_n graphs and we construct co-even domination polynomials and roots of F_n graphs.

2. Co-Even Domination Polynomial Of F_n

We denote the family of co-even dominating sets of fan graph F_n with cardinality i by $D_{coe}(F_n, i)$. We will investigate the co-even dominating sets of F_n graph as follows.

Definition 2.1 Let G be a simple graph of order n and $D_{coe}(F_n, i)$ be the family of co-even dominating sets with cardinality i and $d_{coe}(F_n, i) = |D_{coe}(F_n, i)|$, then

$$D_{coe}(F_n, x) = \sum_{i=\gamma_{coe}(F_n)}^{n} d_{coe}(F_n, i)\, x^i$$

is a co-even domination polynomial of F_n.

Lemma 2.1 Let F_n be the Fan graph with $n + 1$ vertices. Then $\gamma_{coe}(F_n) = n - 2$ when n is even and $\gamma_{coe}(F_n) = n - 1$ when n is odd.

Lemma 2.2 Let F_n be the Fan graph with $n + 1$ vertices and $D_{coe}(F_n, i)$ be the family of all co-even dominating sets with cardinality i. Then

i. $D_{coe}(F_n, i) \neq \phi$ if and only if $n - 2 \leq i \leq n$. Also, $D_{coe}(F_n, i) = \phi$ if and only if $i < n - 2$ or $i > n$, when n is even.

ii. $D_{coe}(F_n, i) \neq \phi$ if and only if $n - 1 \leq i \leq n$. Also, $D_{coe}(F_n, i) = \phi$ if and only if $i < n - 1$ or $i > n$, when n is odd.

Proof

i. Let $i = n - 2$. By the definition of co-even domination number and lemma 2.1, there is at least one co-even dominating set in $D_{coe}(F_n, i)$ say D. Clearly $D_{coe}(F_n, i) \neq \phi$. Suppose $i = (n - 2) + 1$. Then there exists at least one vertex $u \in V - D$. Add u to D. Since every super set of a co-even dominating set is also a co-even set with cardinality $i = (n - 2) + 1$ that is, $i = n - 1$. In this way we can prove that $D_{coe}(F_n, i) \neq \phi$ if and only if $n - 2 \leq i \leq n$. If $i < n - 2$ then there is no co-even dominating set in $D_{coe}(F_n, i)$. Therefore, when n is even $D_{coe}(F_n, i) \neq \phi$ if and only if $n - 2 \leq i \leq n$. Also, $D_{coe}(F_n, i) = \phi$ if and only if $i < n - 2$ or $i > n$.

ii. Let $i = n - 1$. By the definition of co-even domination number and lemma 2.1, there is at least one co-even dominating set in $D_{coe}(F_n, i)$ say D. Clearly $D_{coe}(F_n, i) \neq \phi$. Suppose $i = (n - 1) + 1$. Then there exists at least one vertex $u \in V - D$. Add u to D. Since every super set of a co-even dominating set is also a co-even set with cardinality $i = (n - 1) + 1$ that is, $i = n$. In this way we can prove that $D_{coe}(F_n, i) \neq \phi$ if and only if $n - 2 \leq i \leq n$. If $i < n - 1$ then there is no co-even dominating set in $D_{coe}(F_n, i)$. Therefore, when n is odd $D_{coe}(F_n, i) \neq \phi$ if and only if $n - 2 \leq i \leq n$. Also, $D_{coe}(F_n, i) = \phi$ if and only if $i < n - 1$ or $i > n$.

Lemma 2.3 Let F_n be the Fan graph with $n + 1$ vertices. Then

i. $D_{coe}(F_n, n) = \{[n]\}$ when n is even.

ii. $D_{coe}(F_n, n) = \{[n - 1]\}$ when n is odd.

Proof

i. Here $i = n$. Therefore, the co-even dominating set contains all the vertices except one vertex of the Fan graph F_n. Therefore there is only one set available in $D_{coe}(F_n, n)$. Thus $D_{coe}(F_n, n) = \{[n]\}$, when n is even.

ii. Here $i = n$. Therefore, the co-even dominating set contains all the vertices except one vertex of the Fan graph F_n. Therefore there is only one set available in $D_{coe}(F_n, n)$. Thus $D_{coe}(F_n, n) = \{[n - 1]\}$, when n is odd.

Lemma 2.4 If $Y \in D_{coe}(F_{n-2}, i-1)$ such that $Y \cup \{x\} \in D_{coe}(F_n, i)$ then $Y \in D_{coe}(F_{n-2}, i)$ when n is even and $i-1 = n$.

Lemma 2.5 For every $n \geq 6$

i) If $D_{coe}(F_{n-1}, i-1) = \phi$, $D_{coe}(F_{n-3}, i-1) = \phi$ then $D_{coe}(F_{n-2}, i-1) = \phi$.

ii) If $D_{coe}(F_{n-1}, i-1) \neq \phi$, $D_{coe}(F_{n-3}, i-1) = \phi$ then $D_{coe}(F_{n-2}, i-1) \neq \phi$.

iii) If $D_{coe}(F_{n-1}, i-1) \neq \phi$, $D_{coe}(F_{n-3}, i-1) = \phi$ and $D_{coe}(F_{n-2}, i-1) = \phi$ then $D_{coe}(F_n, i-1) \neq \phi$.

Proof

iv) Case i: when n is even. Since $D_{coe}(F_{n-1}, i-1) = \phi$ by lemma 2.2 $i-1 < n-3$ or $i-1 > n-1 > n-2$. Since $D_{coe}(F_{n-3}, i-1) = \phi$ by lemma 2.2 $i-1 < n-5$ or $i-1 > n-4 \Rightarrow i-1 < n-4$. Hence, we get $i-1 < n-4$ or $i-1 > n-2$. By lemma 2.2 $D_{coe}(F_{n-2}, i-1) = \phi$.

Case ii: when n is odd. Since $D_{coe}(F_{n-1}, i-1) = \phi$ by lemma 2.2 $i-1 < n-2$ or $i-1 > n-1 > n-2$. Since $D_{coe}(F_{n-3}, i-1) = \phi$ by lemma 2.2 $i-1 < n-4$ or $i-1 > n-3 \Rightarrow i-1 < n-3$. Hence, we get $i-1 < n-3$ or $i-1 > n-2$. By lemma 2.2 $D_{coe}(F_{n-2}, i-1) = \phi$.

ii) Case i: when n is even. Since $D_{coe}(F_{n-1}, i-1) \neq \phi$ by lemma 2.2 $n-3 \leq i-1 \leq n-1$. Since $D_{coe}(F_{n-3}, i-1) = \phi$ by lemma 2.2 $i-1 < n-5$ or $i > n-4$. Suppose $D_{coe}(F_{n-2}, i-1) = \phi$ then by lemma 2.2 $i-1 < n-4$ or $i-1 > n-2 \Rightarrow i-1 < n-3$ which is a contradiction. Therefore $D_{coe}(F_{n-2}, i-1) \neq \phi$.

Case ii: when n is odd. Since $D_{coe}(F_{n-1}, i-1) \neq \phi$ by lemma 2.2 $n-2 \leq i-1 \leq n-1$. Since $D_{coe}(F_{n-3}, i-1) = \phi$ by lemma 2.2 $i-1 < n-4$ or $i > n-3$. Suppose $D_{coe}(F_{n-2}, i-1) = \phi$ then by lemma 2.2 $i-1 < n-3$ or $i-1 > n-2 \Rightarrow i-1 < n-2$ which is a contradiction. Therefore $D_{coe}(F_{n-2}, i-1) \neq \phi$.

iii) Case i: when n is even. Since $D_{coe}(F_{n-1}, i-1) \neq \phi$ by lemma 2.2 $n-3 \leq i-1 \leq n-1$. Since $D_{coe}(F_{n-3}, i-1) = \phi$ by lemma 2.2 $i-1 < n-5$ or $i > n-4$. Since $D_{coe}(F_{n-2}, i-1) = \phi$ by lemma 2.2 $i-1 < n-4$ or $i > n-3$. Suppose $D_{coe}(F_n, i-1) = \phi$ then by lemma 2.2 $i-1 < n-2$ or $i-1 > n \Rightarrow i-1 < n-3$ which is a contradiction. Therefore $D_{coe}(F_n, i-1) \neq \phi$.

Case ii: when n is odd. Since $D_{coe}(F_{n-1}, i-1) \neq \phi$ by lemma 2.2 $n-2 \leq i-1 \leq n-1$. Since $D_{coe}(F_{n-3}, i-1) = \phi$ by lemma 2.2 $i-1 < n-4$ or $i > n-3$. Since $D_{coe}(F_{n-2}, i-1) = \phi$ by lemma 2.2 $i-1 < n-3$ or $i-1 > n-2$. Suppose $D_{coe}(F_n, i-1) = \phi$ then by lemma 2.2 $i-1 < n-1$ or $i-1 < n-2 \Rightarrow i-1 < n-3$ which is a contradiction. Therefore $D_{coe}(F_n, i-1) \neq \phi$.

Lemma 2.6 If $D_{coe}(F_n, i) \neq \phi$. Then $D_{coe}(F_{n-1}, i-1) \neq \phi$, $D_{coe}(F_{n-3}, i-1) = \phi$ and $D_{coe}(F_{n-2}, i-1) = \phi$ if and only if $i = n-k$ for some $k \in \mathbb{N}$.
Proof Since $D_{coe}(F_{n-1}, i-1) \neq \phi$ by lemma 2.2 $n-3 \leq i-1 \leq n-1$. Since $D_{coe}(F_{n-2}, i-1) \neq \phi$ by lemma 2.2 $n-4 \leq i-1 \leq n-2$. Since $D_{coe}(F_n, i-1) \neq \phi$ by lemma 2.2 $n-2 \leq i \leq n$. If $i-1 > n-1$, then $i > n$ is not possible. Take $i-1 \leq n-2$ which gives $i = n-k$ for some $k \in \mathbb{N}$.
Similarly we can prove that if $i = n-k$, then we get $D_{coe}(F_{n-1}, i-1) \neq \phi$, $D_{coe}(F_{n-3}, i-1) = \phi$ and $D_{coe}(F_{n-2}, i-1) = \phi$.

Lemma 2.7 If $D_{coe}(F_{n-1}, i-1) \neq \phi, D_{coe}(F_{n-3}, i-1) = \phi$ and $D_{coe}(F_{n-2}, i-1) = \phi$ then $D_{coe}(F_n, i) = \{3,4, \dots, n-2, n-1, n\}$.
Proof Clearly the set $\{3,4, \dots, n-2, n-1, n\}$ has $n-2$ elements. By the definition of co-even domination and F_n, $\{3,4, \dots, n-2, n-1, n\}$ dominates all the vertices upto 16 for $n = 16$. Proceeding like this we obtain $\{3,4, \dots, n-2, n-1, n\}$ vertices dominates all the other vertices.

The other vertices with the cardinality does not satisfies the definition of co-even domination. Hence $D_{coe}(F_n, i) = \{3,4, \dots, n-2, n-1, n\}$.
Similarly we can prove when n is odd.

Lemma 2.8 Let $D_{coe}(F_n, i)$ be the family of all co-even dominating sets with cardinality i. Let $d_{coe}(F_n, i) = |D_{coe}(F_n, i)|$. For every $n \geq 4$,

i. If $i = n$, for some $k \in \mathbb{N}$, then $d_{coe}(F_n, i) = d_{coe}(F_{n-1}, i-2) + d_{coe}(F_{n-1}, i-1)$, when n is even.

ii. If $i = n$, for some $k \in \mathbb{N}$, then $d_{coe}(F_n, i) = d_{coe}(F_{n-1}, i-2) - d_{coe}(F_{n-1}, i-3)$, when n is odd.

Proof
The proof is obvious.

Using lemma 2.8, we obtain $d_{coe}(F_n, i)$ for $3 \leq n \leq 15$ as shown in table 2.1: $d_{coe}(F_n, i)$, the number of dominating sets of cardinality i.

Lemma 2.9 The following properties hold for the coefficients of $D_{coe}(F_n, i)$; for every $n \geq 3$

iii. $d_{coe}(F_n, n) = 2$, when n is odd

iv. $d_{coe}(F_n, n) = 3$, when n is even

v. $d_{coe}(F_n, n-1) = 1$, when n is odd

vi. $d_{coe}(F_n, n-1) = 3$, when n is even

vii. $d_{coe}(F_n, n-2) = 1$, when n is even.

Table 1: $d_{coe}(F_n, i)$, the number of co-even dominating sets of F_n with cardinality i

n / i	1	2	3	4	5	6	7	8	9	10	11	12
3	0	1	3	3								
4	0	1										
5	0	1	5	10	10	5						
6	0	1										
7	0	1	7	21	35	35	21	7				
8	0	1										
9	0	1	9	36	84	126	126	84	36	9		
10	0	1										
11	0	1	11	55	165	330	462	462	330	165	55	11
12	0	1										

3. Conclusion

In this paper we analyse and obtain the co-even dominating sets and co-even domination polynomials of Fan Graph F_n.

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